THE UNIVERSITY OF ALBERTA

STUDIES IN COMBINATORIAL ANALYSIS

by

© DENIS HANSON

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "STUDIES IN COMBINATORIAL ANALYSIS", submitted by DENIS HANSON in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

ABSTRACT

This thesis is devoted to a study of a variety of problems in combinatorial analysis.

Chapter I is devoted to a study of the following problem of Erdös and Rado: Let $\mathcal{F}$ be a family of n-sets, no k of which have pairwise the same intersection. What is the maximal cardinality, $\phi(n,k)$, of all such families $\mathcal{F}$? The upper bound of Erdös and Rado, namely $\phi(n,k) \leq n!(k-1)^n$ is improved by a factor of approximately $\frac{1}{2^n}$. The value of $\phi(2,k)$ is determined and new lower bounds for $\phi(n,3)$ and $\phi(n,4)$ are obtained.

The problem of Erdös and Rado has an application to the following question in number theory: Denote by $f(n)$ the largest number of integers that can be selected from $\{1,2,\ldots,n\}$, no three with pairwise the same greatest common divisor. It is shown that $f(n) < n^{\frac{1}{2}+\varepsilon}$ for every $\varepsilon > 0$, provided $n \geq n_o(\varepsilon)$. This improves the best previous result of $f(n) < n^{\frac{3}{4}+\varepsilon}$ obtained by Erdös in 1964.

By an (n,h,ℓ) graph is meant a graph with n vertices, each of degree at most ℓ and in which no set of independent edges has cardinality greater than h. Let $f(n,h,\ell)$ denote the maximum number of edges that any (n,h,ℓ) graph can contain. $f(n,h,\ell)$ is determined for all values of n, h and ℓ except those for which $\ell \leq 2h$, ℓ odd, and in this case sharp upper and lower bounds are obtained. Let $f(h,\ell) = \max_n f(n,h,\ell)$. The problem of determining $\phi(2,k)$ now becomes that of evaluating $f(k-1,k-1)$ and this is evaluated exactly. This result has recently been obtained by N. Sauer.

In Chapter II we consider some generalizations of the following problem of Schur: What is the largest integer $f(n)$ for which there exists some way of partitioning $\{1,2,\ldots,f(n)\}$ into n sets, no set containing a solution of the equation $x_1 + x_2 = x_3$?

Let $f(n)$ be the largest integer for which there exists some way of partitioning $\{1,2,\ldots,f(n)\}$ into n sets, no set containing a solution to the equation $\sum_{i=1}^{m} a_i x_i = 0$ where the a_i's are given non-zero integers with the property that some proper subset of the a_i's has zero sum. Our main result is an estimate of the lower bound for $f(n)$ and this result is then used to obtain a lower bound for $f(n)$ for a number of special equations, in particular the equation $x_1 + x_2 - x_3 = 0$ of Schur. The system of equations $x_{i,j} + x_{j,j+1} = x_{i,j+1}$, $1 \leq i < j \leq k-1$ is considered in some detail and results of the estimates of $f(n)$ for this system are used to obtain some new estimates for certain Ramsey numbers.

Chapter III is devoted to the so-called property $\mathcal{B}$ of a family of sets. A family $\mathcal{F}$ of sets is said to have property $\mathcal{B}$ if there exists a set $B \subset \cup \mathcal{F}$ such that $B \cap F \neq \emptyset$ and $B \not\supset F$ for all $F \in \mathcal{F}$. Erdös has raised the following question: If n and N are integers, $N \geq 2n-1$, what is the least integer $m_N(n)$ for which there exists a family $\mathcal{F}$ of $m_N(n)$ n-subsets of a set of N elements which does not have property $\mathcal{B}$, but every proper subfamily of $\mathcal{F}$ has property $\mathcal{B}$? Erdös pointed out that $m_N(2) = N$ if N is odd and that $m_N(2)$ does not exist if N is even. Here it is shown that $m_N(n)$ exists for all other values of n and N and some upper bounds are obtained.

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CHAPTER I

A PROBLEM OF ERDÖS AND RADO

§1.1 Introduction

Let n and k be positive integers where $k \geq 3$. Denote by $\phi(n,k)$ the least positive integer such that if $\mathfrak{F}$ is any family of more than $\phi(n,k)$ sets, each of order n, then some k members of $\mathfrak{F}$ have pairwise the same intersection.

P. Erdös and R. Rado [15] proved that for each pair of positive integers n and k, $k \geq 3$, that $\phi(n,k)$ exists. They observed that

$$\phi(1,k) = k-1 \tag{1.1.1}$$

and proved for $n \geq 2$

$$\phi(n,k) \leq n(k-1)\phi(n-1,k) - (k-1)(n-1) \quad . \tag{1.1.2}$$

From (1.1.1) and (1.1.2) Erdös and Rado deduced that

$$\phi(n,k) \leq n!(k-1)^n \left\{ 1 - \sum_{t=1}^{n} \frac{t}{(t+1)!(k-1)^t} \right\} , \tag{1.1.3}$$

and this is the best upper bound that has been obtained up to the present time. On the other hand they proved

$$\phi(n,k) \geq (k-1)^n \quad . \tag{1.1.4}$$

A lower bound for $\phi(n,k)$ which is somewhat better than (1.1.4) can be

found in [1].

The problem of evaluating $\phi(n,k)$ appears to be very difficult. The only values of $\phi(n,k)$ appearing in the earlier literature, except those given by (1.1.1) are $\phi(2,3) = 6$, $\phi(2,4) = 10$ and $\phi(3,3) = 20$. (See [4] and [15].)

In this chapter we evaluate $\phi(2,k)$ for all $k \geq 3$, we obtain an improved upper bound for $\phi(n,k)$, a new lower bound for $\phi(n,3)$ and $\phi(n,4)$ and discuss an application to a problem in number theory.

§1.2 A problem in graph theory and the evaluation of $\phi(2,k)$

The problem of evaluating $\phi(2,k)$ can be formulated in the language of graph theory: Let n, h and ℓ be positive integers. By an (n,h,ℓ) graph we shall mean a graph in which

(a) there are n vertices,

(b) no set of independent edges has cardinality greater than h,

(c) each vertex has degree at most ℓ.

By an (h,ℓ) graph we shall mean a graph satisfying (b) and (c). It is not difficult to see that the problem of determining $\phi(2,k)$ is the same as that of determining the maximum number of edges a $(k-1,k-1)$ graph can contain. Let $E(G)$ denote the number of edges of a graph G and let $f(n,h,\ell) = \max E(G)$ where the maximum is taken over all (n,h,ℓ) graphs G. Our main result is the following theorem:

Theorem 1.2.1

(A) Let $\ell \leq 2h$ and let $h = b[\frac{\ell+1}{2}] + c$, $0 \leq c < [\frac{\ell+1}{2}]$. Then if

$n = 2h+d$, $0 \leq d < b$, we have

$$f(n,h,\ell) = h\ell + d[\tfrac{\ell}{2}] \quad \text{if } \ell \text{ is even}$$

and

$$h\ell + d[\tfrac{\ell}{2}] \leq f(n,h,\ell) \leq h\ell + [\tfrac{d\ell}{2}] \quad \text{if } \ell \text{ is odd} .$$

If $n \geq 2h+b$ then

$$f(n,h,\ell) = h\ell + b[\tfrac{\ell}{2}] \quad .$$

(B) If $n-h \geq \ell > 2h$ then

$$f(n,h,\ell) = h\ell \quad .$$

(C) If $\ell = n-h+s > 2h$, where $1 \leq s \leq h-1$, then

$$f(n,h,\ell) = h(n-h) + [\tfrac{hs}{2}] \; .$$

Before proving Theorem 1.2.1 we introduce some additional terminology and prove some preliminary lemmas.

Let $\mathcal{I} = \{I_1, I_2, \ldots, I_m\}$ be a set of independent edges in a graph G and let $I_j = (x_j, y_j)$ for $j = 1,2,\ldots,m$. We will say a node of G is <u>unsaturated</u> if it is not in the collection $\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\}$ and call an edge <u>unsaturated</u> if it is adjacent to an unsaturated node. For convenience we will call the edges of $\mathcal{I}$ <u>dark</u> and all others <u>light</u>. An <u>alternating path</u> (<u>circuit</u>) in a graph G is a simple path (circuit) such that consecutive edges are not both light or dark. A set of independent edges is said to be <u>maximal</u> in G if

there is no set of independent edges of greater cardinality.

Lemma 1.2.1 In any graph G, a set of independent edges, $\mathcal{J}$, is maximal if and only if G contains no alternating path joining two unsaturated nodes of G.

Proof: If G contains such a path, say $\{p_1,p_2,\ldots,p_{2s}\}$, where the edges $(p_1,p_2),(p_3,p_4),\ldots,(p_{2s-1},p_{2s})$ are light and the edges $(p_2,p_3),(p_4,p_5),\ldots,(p_{2s-2},p_{2s-1})$ are dark, then the set of edges $\{(p_1,p_2),(p_3,p_4),\ldots,(p_{2s-1},p_{2s})\} \cup \{\mathcal{J} - \{(p_2,p_3),(p_4,p_5),\ldots,(p_{2s-2},p_{2s-1})\}\}$ is a larger set of independent edges. Hence $\mathcal{J}$ is not maximal.

Consider a set J of $|\mathcal{J}| + 1$ independent edges, and let J^* be the set of edges of J which are not in $\mathcal{J}$. We may assume that it is impossible to form an alternating circuit from the edges of $\mathcal{J}$ and the edges of J^*. Suppose β edges in J^* are unsaturated and $\alpha = |J^*| - \beta$ are not. Then, by the remark just made, at least $\alpha+1$ edges in $\mathcal{J}$ do not belong to J. A moment reflection shows that $\beta = 2$ and that one can construct an alternating path starting with one of these unsaturated edges and terminating with the other. This completes the proof.

In what follows we shall sometimes have occasion to modify a graph G by deleting edges or by adding new vertices and edges, thereby obtaining a new graph G'. Since the terms unsaturated nodes, unsaturated edges, light edges, dark edges and alternating path are defined for G in terms of some particular set $\mathcal{J}$ of independent edges, we wish to point out that in constructing G' from G we shall not alter $\mathcal{J}$, that is,

the edges in $\mathcal{I}$ are also edges of G' and the terms mentioned above are defined for G' with respect to $\mathcal{I}$. We shall also henceforth assume that $\mathcal{I}$ is maximal in G.

Partition the edges of $\mathcal{I}$ into classes as follows:

$\mathcal{A} = \{I_j \mid I_j \in \mathcal{I}$ and either x_j or y_j belongs to at least two unsaturated edges$\}$

$\mathcal{B} = \{I_j \mid I_j \in \mathcal{I}$ and both x_j and y_j are joined to an unsaturated node$\}$

$\mathcal{C} = \{I_j \mid I_j \in \mathcal{I} - \mathcal{A} - \mathcal{B}$ and either x_j or y_j belongs to an unsaturated node$\}$

$\mathcal{D} = \{I_j \mid I_j \in \mathcal{I} - \mathcal{A} - \mathcal{B} - \mathcal{C}\}$.

It is easy to see that $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| + |\mathcal{D}| = |\mathcal{I}|$.

To each $I_j \in \mathcal{I}$ we assign a <u>weight</u>, $w(I_j)$ as follows:

$$\begin{aligned} w(I_j) &= (\text{number of unsaturated edges adjoined to } I_j) \\ &\quad + \tfrac{1}{2}\,(\text{number of remaining edges adjacent to } x_j \text{ and } y_j)\,. \end{aligned}$$

It is clear that, since $\mathcal{I}$ is maximal,

$$E(G) = \sum_{I \in \mathcal{I}} w(I)\,.$$

<u>Lemma 1.2.2</u> Let G be an (n,h,ℓ) graph with maximal set of independent edges $\mathcal{I}$. Then there exists an (h,ℓ) graph G' such that

$$E(G) = E(G')$$

and

$$w'(I) \leq \ell+1$$

for all $I \varepsilon \mathcal{I}$, where $w'(I)$ is the weight of I in G' .

Proof: It is clear that any edge belonging to $\mathcal{B}$, $\mathcal{C}$ or $\mathcal{D}$ has weight at most $\ell+1$, $\ell+\frac{1}{2}$ or ℓ respectively. Hence if $\mathcal{A}$ is empty we may take $G' = G$. Therefore assume $\mathcal{A}$ is non-empty. We will induct on h . If $h = 1$ then G is a star and again we may take $G' = G$. Assume that the lemma is true for all $(n,h-1,\ell)$ graphs. Let $I_1 \varepsilon \mathcal{A}$, where $I_1 = (x_1,y_1)$, and assume that x_1 is the vertex adjacent to the unsaturated nodes. Let S be the set of light saturated edges incident with x_1 , T the set of light edges incident with y_1 and R the set of unsaturated nodes adjacent to x_1 . Now modify the graph G as follows:

(1) Replace each edge in S by a new edge joining x_1 to a new vertex.

(2) If $(y_1,z_1),(y_1,z_2),\ldots,(y_1,z_k)$ are the edges of T , replace these by $(z_1,w),(z_2,w),\ldots,(z_k,w)$ where w is a new vertex.

(3) If $v \varepsilon R$ and v is adjacent to the saturated nodes $t_1,t_2,\ldots,t_m$ then replace the edges $(v,t_1),(v,t_2),\ldots,(v,t_m)$ by $(u,t_1),(u,t_2),\ldots,(u,t_m)$ where v is a new vertex.

It is easy to see that t_i and z_j are not the end points of an edge in $\mathcal{I}$, since otherwise $\mathcal{I}$ would not be maximal in G . By Lemma 1.2.1 it follows that $\mathcal{I}$ is a maximal collection of independent

edges in the resulting graph G_1 . Thus we have a graph G_1 consisting of a star, G_2 , containing I_1 , and an $(h-1,\ell)$ graph, G_3 , such that $E(G_1) = E(G)$. By the induction hypothesis the conclusion of the lemma holds for G_3 . To complete the argument set $G' = G_2 \cup G_3'$.

Lemma 1.2.3 Let G be an (n,h,ℓ) graph with a maximal set of independent edges $\mathcal{I}$ and let G' be defined by Lemma 1.2.2. If G' contains a component with k independent edges, where $k \leq [\frac{\ell-1}{2}]$, then the weight of any independent edge in this component is at most ℓ .

Proof: By Lemma 1.2.2, if any independent belongs to class $\mathcal{A}$ it has weight at most ℓ by construction. Any other independent edge has weight at most $2 + \frac{1}{2}(2k-1 + 2k-1) = 2k+1 \leq \ell$.

Note that Lemma 1.2.3 will apply to Theorem 1.3.1 whenever $\ell > 2h$. The following lemma will apply in the case $\ell \leq 2h$.

Lemma 1.2.4 Let G be an (n,h,ℓ) graph with a maximal set of independent edges $\mathcal{I}$ and let G' be defined by Lemma 1.2.2. Suppose that there exists a pair of edges $I_1 = (x_1,y_1)$ and $I_2 = (x_2,y_2)$ in $\mathcal{I}$ and two unsaturated nodes, x_1' and x_2' , adjacent to x_1 and x_2 respectively and suppose that the edges I_1 and I_2 belong to the same component of G' . Then there exists a graph G'' which has all the properties of G' required by Lemma 1.2.2 and in which I_1 and I_2 are in distinct components.

Proof: We may assume that among all such pairs of edges of $\mathcal{I}$, I_1 and I_2 are chosen such that the path joining x_1' and x_2' is of minimal

length. Consider any path $P(x_1',x_2') = \{x_1' = p_0,p_1,\ldots,p_k,p_{k+1} = x_2'\}$ joining x_1' and x_2' and all alternating paths $Q(x_1',p_i) = \{x_1',q_1,\ldots,q_t,p_i\}$ such that none of the nodes $p_{i+1},p_{i+2},\ldots,p_{k+1}$ belong to $Q(x_1',p_i)$. That such paths exist follows from the maximality of $\mathcal{J}$ and Lemma 1.2.1. Let m be the maximum i, $1 \leq i \leq k$, for which such a path exists, i.e. let $Q(x_i,p_m) = \{x_1',q_1,q_2,\ldots,q_t,p_m\}$ be an alternating path that does not pass through any of the nodes $p_{m+1},p_{m+2},\ldots,p_{k+1}$ and such that m is maximal in this respect. By the minimality condition imposed on $P(x_1',x_2')$ it follows that the only unsaturated nodes belonging to $P(x_1',x_2')$ are the nodes x_1' and x_2'.

<u>Case 1</u> $m < k$ and $q_t = p_{m-1}$.

Since $q_t = p_{m-1}$, the edge (p_{m-1},p_m) is light, for otherwise there exists an alternating path $Q(x_1',p_{m+1})$ contradicting the maximality of m. Similarly we may assume that (p_m,p_{m+1}) is a light edge. Let (p,p_m) be a member of $\mathcal{J}$. Replace the edge (p_m,p_{m+1}) by a new edge (p_m,z_o) where z_o is a new node. It is clear that the resulting graph has the required properties with regard to number of edges and maximal degree of any node. We now consider the possibility that the resulting graph contains an alternating path $R(z_o,z) = \{z_o,p_m,p,\ldots,z\}$ where z is an unsaturated node. Assume such a path exists.

If $z \neq x_1'$ we must have that $R(z_o,z)$ and $Q(x_1',p_m)$ have at least one edge in common, for otherwise G' contained an alternating path $\{x_1',q_1,\ldots,p_{m-1},p_m,p,\ldots,z\}$ and this is impossible by the maximality of $\mathcal{J}$. Since $R(z_o,z)$ and $Q(x_1',p_m)$ are both alternating paths, the last edge of $R(z_o,z)$ common to $Q(x_1',p_m)$ must be dark.

Assume this edge is (q_j,q_{j+1}) for some j, $1 \leq j \leq t-1$ and that $R(z_o,z) = \{z_o,p_m,p,\ldots,q_j,q_{j+1},r,\ldots,z\}$. Then the path $S(x_1',z) = \{x_1',q_1,\ldots,q_j,q_{j+1},r,\ldots,z\}$ is an alternating path joining two distinct unsaturated nodes in G' and we have a contradiction. If on the other hand $R(z_o,z) = \{z_o,p_m,p,\ldots,r,q_{j+1},q_j,\ldots,z\}$ then there exists an alternating path $S(x_1',p_w) = \{x_1',q_1,\ldots,q_j,q_{j+1},r,\ldots,p_w\}$ where $w \geq m+1$. For example, if the sub-path $R(z_o,q_j) = \{z_o,p_m,p,\ldots,r,q_{j+1},q_j\}$ of $R(z_o,z)$ does not contain any of the nodes $p_{m+1},p_{m+2},\ldots,p_{k+1}$ then we could choose $S(x_1',p_w) = S(x_1',p_{m+1}) = \{x_1',q_1,\ldots,q_j,q_{j+1},r,\ldots,p,p_m,p_{m+1}\}$ and if $R(z_o,q_j)$ contains a node p_w, $w \geq m+1$, then we could choose $S(x_1',p_w) = \{x_1',q_1,\ldots,q_j,q_{j+1},r,\ldots,p_w\}$ and in both cases we have a contradiction to the maximality of m. If $z = x_1'$, then there exists an alternating path $Q(x_1',p_{m+1}) = \{x_1',\ldots,p,p_m,p_{m+1}\}$ or an alternating path $Q(x_1',p_w)$, $w > m+1$, depending on the structure of the path $R(z_o,z)$ as above in the case $z \neq x_1'$. Therefore we conclude in both cases that no such path $R(z_o,z)$ exists.

<u>Case 2</u> $\quad m < k$ and $q_t \neq p_{m-1}$.

By similar reasoning to that used in Case 1 we must have that the edges (q_t,p_m) and (p_m,p_{m+1}) are light edges. Let (p,p_m) be a member of $\mathcal{J}$. Replacing the edge (p_m,p_{m+1}) by a new edge (p_m,z_o) where z_o is a new node it is easy to see by the same argument used in Case 1 that our graph has the required properties with regard to number of edges, number of independent edges and maximal degree of any node.

<u>Case 3</u> $\quad m = k$ and $q_t = p_{k-1}$.

Since $q_t = p_{k-1}$, the edge (p_{k-1},p_k) is a light edge. Replace the edge (p_{k-1},p_k) by a new edge (p_k,z_o) where z_o is a new node. Let (p,p_k) be a member of $\mathcal{J}$. As in the previous cases we must show that the resulting graph does not contain an alternating path $R(z_o,z) = \{z_o,p_k,p,\ldots,z\}$ where z is an unsaturated node. If such a path exists and $z \neq x_2'$ then G' contained an alternating path $S(z,x_2') = \{z,\ldots,p,p_k,p_{k+1} = x_2'\}$. If $z = x_2'$, by similar reasoning to the cases $m < k$, $z \neq x_1'$, we can deduce the existence of an alternating path joining x_1' and x_2' . In both possibilities we have a contradication, hence no such path $R(z_o,z)$ exists.

<u>Case 4</u> $m = k$ and $q_t \neq p_{k-1}$.

It is clear that (q_t,p_k) is a light edge. Replace the edge (q_t,p_k) by a new edge (p_k,z_o) where z_o is a new node. Similar arguments to those used in the previous cases then imply that there is no alternating path $R(z_o,z)$, where z is an unsaturated node, in the resulting graph. Treating all such alternating paths $Q(x_1',p_k)$ where $q_t \neq p_{k-1}$ in the same manner the resulting graph has the property that any alternating path $Q(x_1',p_m)$ as described before satisfies either $m < k$ or $m = k$ and $q_t = p_{k-1}$.

The effect of the above procedure given in Cases 1 through 4 is that the resulting graph no longer contains the path $P(x_1',x_2')$. Repeating this procedure on all possible choices of the path $P(x_1',x_2')$ we obtain a graph G'' in which x_1' and x_2' are in distinct components, which completes the proof of the lemma.

In the proof of Theorem 1.2.1 we will use the following lemma due to Dirac [8] to construct graphs with $f(n,h,\ell)$ edges.

Lemma 1.2.5 Let G be a graph with n nodes and let $d(a)$ be the degree of any node a in G. If $d(a) + d(b) \geq n$ for any pair of nodes a and b of G, then G contains a Hamiltonian circuit.

We now will prove Theorem 1.2.1. First we will exhibit graphs with $f(n,h,\ell)$ edges and then show that this is the maximum number of edges.

Proof of Theorem 1.2.1

I. Case A $\ell \leq 2h$, $h = b[\frac{\ell+1}{2}] + c$, $0 \leq c < [\frac{\ell+1}{2}]$.

If ℓ is odd and $n = 2h+d$, $0 \leq d \leq b$, consider the complete graph, H, on $m = 2[\frac{\ell+1}{2}] + 2c + 1 = \ell + 2c + 2$ nodes. Note that $m \geq 4c + 3$. By Lemma 1.2.5, H has a Hamiltonian circuit. Remove any such circuit. Every node of the resulting graph has degree $m-3$. Repeat this removal of a Hamiltonian circuit c times and the resulting graph, H_1, has m nodes each of degree $m-1-2c$. This repetition is possible since $2(m-1-2c) \geq m$ whenever $m \geq 4c + 2$.

Now $m-1-2c = \ell+1 \geq \frac{m}{2}$. Therefore H_1 has a Hamiltonian circuit. Let the edges of this circuit be (x_1,x_2), (x_2,x_3), ..., (x_m,x_1). Remove the edges (x_1,x_2), (x_3,x_4), ..., (x_{m-2},x_{m-1}), (x_m,x_1). Note that m is odd, therefore in the resulting graph, H_2, every node has degree ℓ except x_1, which has degree $\ell-1$, and the edges (x_2,x_3), (x_4,x_5), ..., (x_{m-1},x_m) are $[\frac{m}{2}] = [\frac{\ell+1}{2}] + c$ independent edges. That is H_2

is an $(\ell+2c+2,[\frac{\ell+1}{2}]+c,\ell)$ graph and it is easy to see that $E(H_2) = ([\frac{\ell+1}{2}]+c)\ell+[\frac{\ell-1}{2}]$. If we let $c = 0$ in the above construction we obtain a graph H_3 that is an $(\ell+2,[\frac{\ell+1}{2}],\ell)$ graph and $E(H_3) = [\frac{\ell+1}{2}]\ell+[\frac{\ell-1}{2}]$.

Taking $d-1$ distinct H_3 graphs together with H_2 and $b-d$ complete $\ell+1$ graphs we obtain an (n,h,ℓ) graph, G, such that $E(G) = h\ell + d[\frac{\ell}{2}]$.

If ℓ is even we consider a complete $\ell+2c+1$ graph and proceed in a similar manner to the case ℓ odd to obtain a (n,h,ℓ) graph, G, such that $E(G) = h\ell+d[\frac{\ell}{2}]$. Therefore we have that $f(n,h,\ell) \geq h\ell+d[\frac{\ell}{2}]$ for ℓ even and $f(n,h,\ell) \geq h\ell+d[\frac{\ell}{2}]$ for ℓ odd. Observe that if $n > 2h+b$ these results still hold. If ℓ is odd, the trivial example of an $(8,3,1)$ graph shows that we cannot do any better in general. However the following graph illustrates that under certain conditions we can have $E(G) = h\ell + [\frac{d\ell}{2}] > h\ell + d[\frac{\ell}{2}]$:

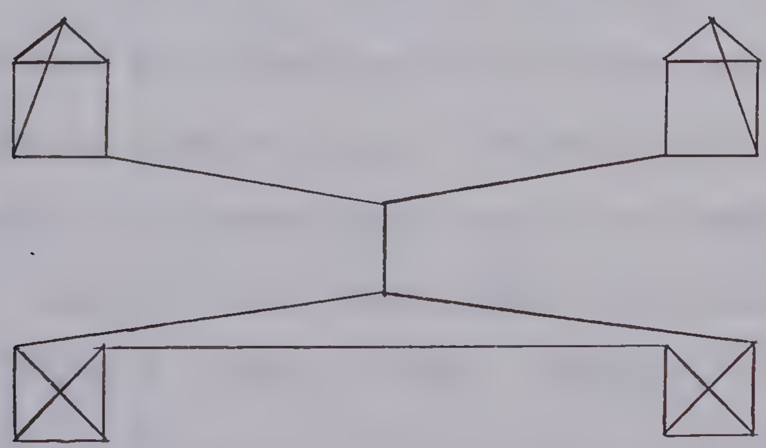

Figure 1.

Figure 1 is a $(20,9,3)$ graph. Note that $b = 4$ and $d = 2$ and that the graph is regular of degree 3. Hence we have
$\frac{1}{2}(20.3) = 30 = h\ell+[\frac{d\ell}{2}] > h\ell+d[\frac{\ell}{2}] = 29$ edges.

Case B $\quad n-h \geq \ell > 2h$.

Let $\mathcal{I} = \{I_1,I_2,\ldots,I_h\}$ be a collection of h independent edges and let $I_j = (x_j,y_j)$ for $i = 1,2,\ldots,h$. Consider a graph consisting of $\mathcal{I}$ and $n-2h$ isolated nodes, z_i, $i = 1,2,\ldots,n-2h$. For each j, $j = 1,2,\ldots,h$, join x_j to $\ell-1$ of the nodes $y_1,y_2,\ldots,y_{j-1},y_{j+1},\ldots,y_h,z_1,z_2,\ldots,z_{n-2h}$. The resulting graph, G, is an (n,h,ℓ) graph consisting of h independent edges of class $\mathcal{A}$ and $E(G) = h\ell$. That is, $f(n,h,\ell) \geq h\ell$.

Case C $\quad \ell = n-h+s$, $\ell > 2h$, $1 \leq s \leq h-1$.

Consider a complete h graph, H_{h-1}, and its edge complementary graph H. H_{h-1} contains a Hamiltonian circuit by Lemma 1.2.5. Remove $[\frac{h+1}{2}]$ of the edges of this circuit where the edges are chosen to be disjoint if h is even and at most one pair of edges adjacent if h is odd. Let the resulting graph be H_{h-2}. Define H_1 to be H together with the $[\frac{h}{2}]$ edges of the Hamiltonian circuit not removed. Define H_{h-3} to be H_{h-1} with all of the Hamiltonian circuit removed and H_2 to be the graph consisting of this circuit. Proceeding in this manner, provided the conditions of Lemma 1.2.5 hold, we can construct a sequence of graphs $H_1,H_2,\ldots,H_{h-1}$ such that $E(H_s) = [\frac{hs}{2}]$ for $s = 1,2,\ldots,h-1$.

Let G be the graph described in Case B when $\ell = n-h$. From

the graph H_s on the nodes $x_1, x_2, \ldots, x_h$ when $\ell = n-h+s$, $s = 1,2,\ldots,h-1$. The resulting graph, G_1, is an (n,h,ℓ) graph such that $E(G_1) = h(n-h) + [\frac{hs}{2}]$. That is, $f(n,h,\ell) \geq h(n-h) + [\frac{hs}{2}]$.

II. By the proceeding lemmas, given an (n,h,ℓ) graph G with a maximal set of independent edges $\mathcal{I}$, we can construct an (h,ℓ) graph G' such that in G'

(i) the maximum weight of any independent edge is $\ell+1$

(ii) no two independent edges belong to the same component of G' if they are adjacent to distinct unsaturated nodes

(iii) any independent edge of class $\mathcal{A}$ has weight at most ℓ and is a component of G'

(iv) the weight of any independent edge in a component of G' containing $k \leq [\frac{\ell-1}{2}]$ independent edges is at most ℓ.

Consider the case $\ell \leq 2h$, $n = 2h+d$ and $0 \leq d < b$, where $h = b[\frac{\ell+1}{2}] + c$, $0 \leq c < [\frac{\ell+1}{2}]$. Clearly

$$f(n,h,\ell) \leq \frac{1}{2}(2h+d)\ell = h\ell + [\frac{d\ell}{2}] ,$$

since every node has degree at most ℓ. By Part I we are done.

Now assume $n \geq 2h+b$. Since $E(G) = E(G')$ we will maximize $E(G')$. Any component of G' containing an unsaturated node and m independent edges contains, by (ii), at most $m\ell + [\frac{\ell}{2}]$ edges. Therefore, by (iv), to maximize $E(G')$, G' must contain as many components, with at least $[\frac{\ell+1}{2}]$ independent edges, as possible. Hence

$$f(n,h,\ell) \leq (b-1)([\frac{\ell+1}{2}]\ell+[\frac{\ell}{2}]) + ([\frac{\ell+1}{2}]+c)\ell + [\frac{\ell}{2}] = h\ell + b[\frac{\ell}{2}]$$

and we are done by Part I.

If $n-h \geq \ell > 2h$, then by (iv)

$$f(n,h,\ell) \leq h\ell$$

and again, we are done by Part I.

Finally consider the case $\ell = n-h+s$, $1 \leq s \leq h-1$. If $\mathcal{I} = \{I_1,I_2,\ldots,I_h\}$ and $I_j = (x_j,y_j)$ for $j = 1,2,\ldots,h$, then it is clear that if $d(x_j)$ exceeds $2h$, I_j is in class $\mathcal{A}$. By Lemma 1.2.1, if $d(x_j) > 2h$ and $d(x_k) > 2h$ for some j and k, $1 \leq j,k \leq h$, then y_j is not adjacent to y_k. Therefore, for each j, $j = 1,2,\ldots,h$,

$$w'(I_j) \leq n - 2h + \frac{1}{2}(\ell-(n-2h)) + \frac{1}{2}h ,$$

where $\frac{1}{2}h$ is the maximum degree of y_j if $d(x_j) > 2h$, i.e.

$$w'(I_j) \leq n - 2h + \frac{1}{2}(2h+s) = n - h + \frac{s}{2} .$$

It now follows that

$$f(n,h,\ell) \leq h(n-h) + [\frac{hs}{2}]$$

and by Part I, the theorem is proved.

<u>Corollary 1.2.1</u> If G is an (n,h,ℓ) graph then

$$E(G) \leq \max\{\binom{2h+1}{2}, h(n-h) + \binom{h}{2}\} ,$$

with equality holding only if G consists of a complete $(2h+1)$-graph and $n-(2h+1)$ isolated nodes or if G consists of a complete h-graph each node of which is also joined to each of the remaining $n-h$ nodes.

Proof: If $\ell \leq 2h$ in Theorem 1.2.1, then since $f(n,h,\ell)$ is non-decreasing as n increases

$$\max_{1\leq \ell \leq 2h} f(n,h,\ell) \leq \max_{1\leq \ell \leq 2h} (h\ell + b[\tfrac{\ell}{2}])$$

where b is defined as before. It is easy to see that this last expression is maximized when $b = 1$, $\ell = 2h$. Therefore

$$E(G) \leq 2h\cdot h+h = \binom{2h+1}{2} \quad .$$

Since $b = 1$ and $\ell = 2h$ we have $c = 0$ in Part I, Case A, and thus G is a complete $(2h+1)$-graph together with $n-(2h+1)$ isolated nodes.

If $\ell < 2h$ in Theorem 1.2.1,

$$\max_{2h<\ell\leq n-1} f(n,h,\ell) = \max_{1\leq s\leq h-1} (h(n-h)+[\tfrac{hs}{2}]) \quad ,$$

that is

$$E(G) \leq h(n-h) + \binom{h}{2} \quad .$$

Taking $s = h-1$ in Part I, Case C, we have H_{h-1}, a complete h-graph, every node of which has degree $n-1$ and the result of the corollary follows.

Note that the $\max\{\binom{2h+1}{2}, h(n-h) + \binom{h}{2}\}$ depends on the sign of $n - 2\frac{1}{2}h - 1\frac{1}{2}$.

The result given by Corollary 1.2.1 has been obtained by J.W. Moon [20].

The following result has previously been obtained by a different method by N. Sauer (to appear).

Corollary 1.2.2

$$\phi(2,k) = \begin{cases} k(k-1) & k \text{ odd} \\ (k-1)^2 + [\frac{k-2}{2}] & k \text{ even} \end{cases} .$$

Proof: As we remarked previously, $\phi(2,k)$ is the maximum number of edges in a $(k-1,k-1)$ graph. If k is odd we have $b = 2$ in Theorem 1.2.1 Part A and if k is even we have $b = 1$ in Theorem 1.2.1 Part A and the result follows.

§1.3 An upper bound for $\phi(3,k)$

Let $\mathcal{F}$ be a family of $\phi(3,k)$ sets, each of order 3, no k of which have pairwise the same intersection. Without loss of generality assume that the family $\mathcal{F}$ contains the sets

$$F_i = \{3i-2,3i-1,3i\} \quad , \quad i = 1,2,\ldots,k-1 .$$

For $j = 1,2,\ldots,3k-3$ let

$$\mathcal{A}_j = \{F \mid F \in \mathcal{F} , j \in F, F \neq F_i, i = 1,2,\ldots,k-1\}$$

and for $j = 1,2,\ldots,k-1$ let

$$\mathcal{B}_j = \mathcal{A}_{3j-2} \cup \mathcal{A}_{3j-1} \cup \mathcal{A}_{3j}$$

where each set in $\mathcal{B}_j$ is counted according to its multiplicity.

Definition: A set $F \in \mathcal{F}$ is $\mathcal{F}$-disjoint if and only if $F \in \mathcal{B}_i$ for some i, $1 \leq i \leq k-1$ and $F \notin \mathcal{B}_j$, $j \neq i$.

Lemma 1.3.1 If $k \geq 6$, then the maximum number of distinct $\mathcal{F}$-disjoint sets in any $\mathcal{B}_i$, $i = 1,2,\ldots,k-1$, is at most $\phi(2,k)-1$.

Proof: It is clear that we need only consider $\mathcal{B}_1$. Assume the set $T = \{1,x,y\}$ for some x and y is a $\mathcal{F}$-disjoint set belonging to $\mathcal{A}_1$. There are two possibilities:

(i) every $\mathcal{F}$-disjoint set belonging to $\mathcal{B}_1$ belongs to $\mathcal{A}_1$

(ii) there exists at least one $\mathcal{F}$-disjoint set in $\mathcal{A}_2$ and/or $\mathcal{A}_3$ that does not belong to $\mathcal{A}_1$.

If (i) holds, since no element can belong to more than $\phi(2,k)$ sets without there existing k of them with pairwise the same intersection, we are done.

If (ii) holds then any $\mathcal{F}$-disjoint set in $\mathcal{A}_2$ or $\mathcal{A}_3$ must have non-empty intersection with the set T. Therefore there are at most two $\mathcal{F}$-disjoint sets, $T_1 = \{1,u_1,v_1\}$ and $T_2 = \{1,u_2,v_2\}$, where u_1, u_2, v_1 and v_2 are distinct elements, belonging to $\mathcal{A}_1$ and not to $\mathcal{A}_2$ or $\mathcal{A}_3$. Then the maximum number of $\mathcal{F}$-disjoint sets belonging to $\mathcal{A}_1$ and not to $\mathcal{A}_2$ or $\mathcal{A}_3$ is the maximum number of pairs of elements, (a,b), that a collection of 2-sets can contain such that no three are pairwise disjoint and no k have pairwise the same intersection. It follows from Theorem 1.2.1 that there are at most $2k$ such sets. If T_1 and T_2 belong to $\mathcal{A}_1$ then there are at most four $\mathcal{F}$-disjoint sets in $\mathcal{A}_2$ or $\mathcal{A}_3$ that do not also belong to $\mathcal{A}_1$, and since there are

at most $k-1$ $\mathcal{F}$-disjoint sets belonging to $\mathcal{A}_1$ and $\mathcal{A}_2$ or $\mathcal{A}_1$ and $\mathcal{A}_3$ we are done by Corollary 1.2.2 of Theorem 1.2.1. If there are not two distinct sets of the form T_1 and T_2 belonging to any of $\mathcal{A}_1$, $\mathcal{A}_2$ or $\mathcal{A}_3$ the result follows in a similar manner.

It is clear that Lemma 1.3.1 is not as strong as possible with respect to the restriction on k. However, since the upper bound to be obtained for $\phi(3,k)$ will rely on the following lemma, any strengthening of Lemma 1.3.1 would be of no advantage in this direction.

<u>Lemma 1.3.2</u> If for some distinct i, j and ℓ, $1 \leq i,j,\ell \leq k-1$, $\mathcal{A}_{3i-2}$, $\mathcal{A}_{3j-2}$ and $\mathcal{A}_{3\ell-2}$ each contain at least five $\mathcal{F}$-disjoint sets, distinct in the sense that their pairwise intersections in $\mathcal{A}_{3i-2}$, $\mathcal{A}_{3j-2}$ and $\mathcal{A}_{3\ell-2}$ are the elements $3i-2$, $3j-2$ and $3\ell-2$ respectively, then $\mathcal{F}$ does not contain any sets of the form:

$$\begin{array}{llll} \{3i-1,3j-1,x\}, & \{3i-1,3j,x\}, & \{3i-1,3\ell-1,x\}, & \{3i-1,3\ell,x\} \\ \{3i,3j-1,x\}, & \{3i,3j,x\}, & \{3i,3\ell-1,x\}, & \{3i,3\ell,x\} \\ \{3j-1,3\ell-1,x\}, & \{3j-1,3\ell,x\}, & \{3j,3\ell-1,x\}, & \{3j,3\ell,x\} \end{array}$$

where (i) $x \notin \{1,2,\ldots,3k-3\}$

or where (ii) $x \in \{3i-1,3i,3j-1,3j,3\ell-1,3\ell\}$.

<u>Proof:</u> (i) Assume there is a set $T = \{3i-1,3j-1,x\}$ in $\mathcal{F}$. Then it is easy to see that we can choose a pair of disjoint sets from $\mathcal{A}_{3i-2}$ and $\mathcal{A}_{3j-2}$ respectively, such that neither contains the element x. But then these sets together with the sets $T,F_1,F_2,\ldots,F_{i-1},F_{i+1},\ldots,F_{j-1},F_{j+1},\ldots,F_{k-1}$ consitute a collection of k pairwise disjoint sets

in $\mathcal{F}$. Clearly any other set belonging to case (i) may be handled accordingly.

(ii) Assume there is a set $T = \{3i-1, 3j-1, 3\ell-1\}$ in $\mathcal{F}$. Let $\{A_1^i, A_2^i, A_3^i, A_4^i, A_5^i\}$, $\{A_1^j, A_2^j, A_3^j, A_4^j, A_5^j\}$ and $\{A_1^\ell, A_2^\ell, A_3^\ell, A_4^\ell, A_5^\ell\}$ be collections of five $\mathcal{F}$-disjoint sets in $\mathcal{A}_{3i-2}$, $\mathcal{A}_{3j-2}$ and $\mathcal{A}_{3\ell-2}$ respectively where the A_v^u's satisfy the conditions of the lemma. Since the intersection of any two A_s^j's , $s = 1,2,\ldots,5$, is $3j-2$, any set A_t^i , $t = 1,2,\ldots,5$, is disjoint from at least three A_s^j's . In turn every A_r^ℓ , $r = 1,2,\ldots,5$, is disjoint from at least one of these three A_s^j's and similarly at least one of the A_r^ℓ's is disjoint from the set A_t^i chosen originally. Hence, we can find three pairwise disjoint sets say A_1^i, A_1^j and A_1^ℓ . But now the sets $T, A_1^i, A_1^j, A_1^\ell, F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_{j-1}, F_{j+1}, \ldots, F_{\ell-1}, F_{\ell+1}, \ldots, F_{k-1}$ constitute a collection of k pairwise disjoint sets in $\mathcal{F}$. All other possibilities under case (ii) may be treated accordingly.

The following example shows that Lemma 1.3.2 is no longer valid if five is replaced by four: Let a, b, c, d, e, f, g and h be eight distinct elements, none of which belong to any F_i , $i = 1,2,\ldots,k-1$. Now let the $\mathcal{F}$-disjoint sets satisfying the conditions of the lemma be

$$\{3i-2,a,b\},\quad \{3i-2,c,d\},\quad \{3i-2,e,f\},\quad \{3i-2,g,h\}$$

$$\{3j-2,e,g\},\quad \{3j-2,f,h\},\quad \{3j-2,a,c\},\quad \{3j-2,b,d\}$$

$$\{3\ell-2,e,h\},\quad \{3\ell-2,f,g\},\quad \{3\ell-2,a,d\},\quad \{3\ell-2,b,c\} .$$

It is not possible to choose three pairwise disjoint sets from the above collection as would be necessary in the proof of the lemma.

Lemma 1.3.3 For those i, $1 \leq i \leq k-1$, such that none of the sets $\mathcal{A}_{3i-2}$, $\mathcal{A}_{3i-1}$ or $\mathcal{A}_{3i}$ contain at least five $\mathcal{F}$-disjoint sets, distinct in the sense of Lemma 1.3.2, the maximum number of sets in $\mathcal{B}_i$ which belong to one and only one $\mathcal{A}_j$, $j = 1,2,\ldots,3k-3$, is at most $4k$.

Proof: Consider any such i. Assume $\mathcal{A}_{3i-2}$ contains t $\mathcal{F}$-disjoint sets, distinct in the sense of Lemma 1.3.2, where $1 \leq t \leq 4$. Further assume neither $\mathcal{A}_{3i-1}$ nor $\mathcal{A}_{3i}$ contains more than t such sets. If $t = 3$ or 4, the result follows by the arguments used in the proof of Lemma 1.3.1. The same arguments show that if $t = 2$ the maximum number of sets of the desired type is at most $2k+4$. If $t = 1$, then each of $\mathcal{A}_{3i-2}$, $\mathcal{A}_{3i-1}$ and $\mathcal{A}_{3i}$ can contain at most $k-1$ sets of the desired type and this completes the proof of the lemma.

We are now in a position to obtain an upper bound for $\phi(3,k)$, $k \geq 6$. We will obtain this upper bound by counting the maximum number of sets in a family $\mathcal{F}$ of $\phi(3,k)$ 3-sets in two ways and maximizing each with respect to the other.

Theorem 1.3.1 $\phi(3,k) \leq \frac{9}{5}(k-1)\phi(2,k) + \frac{4}{5}(k-1)^2$, $k \geq 6$.

Proof: For some t, $1 \leq t \leq k-1$, assume that t of the collections $\mathcal{B}_i$, $i = 1,2,\ldots,k-1$, contain at least five $\mathcal{F}$-disjoint sets, distinct in the sense of Lemma 1.3.2, in at least one of the collections $\mathcal{A}_{3i-2}$, $\mathcal{A}_{3i-1}$ or $\mathcal{A}_{3i}$. We may assume without loss of generality that these $\mathcal{B}_i$'s are $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_t$ and that the $\mathcal{A}_i$'s are $\mathcal{A}_1, \mathcal{A}_4, \ldots, \mathcal{A}_{3t-2}$.

By the preceding lemmas we have that any set that is not $\mathcal{F}$-disjoint appears at least twice in the collection of all the $\mathcal{B}_i$'s when repetitions are counted. Also, $\mathcal{B}_i$, $i > t$, contains at most $4k$ sets which belong to exactly one of the collections $\mathcal{A}_j$, $j = 1,2,\ldots,3k-3$. Therefore, by Lemma 1.3.1 we have

$$(1.3.1) \qquad \phi(3,k) \leq (k-1) + t\,(\phi(2k,k)-1) + t\,\frac{2(\phi(2,k)-1)}{2} + (k-1-t)\,\frac{3(\phi(2,k)-1)}{2} + (k-1-t)\,\frac{4k}{2} .$$

On the other hand, Lemma 1.3.2 and Lemma 1.3.3 imply that any set in $\mathcal{F}$ belonging to $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_5, \mathcal{A}_6, \ldots, \mathcal{A}_{3t-1}$ or $\mathcal{A}_{3t}$ also belongs to some other $\mathcal{A}_j$, $j > 3t$. Therefore it follows that

$$(1.3.2) \qquad \phi(3,k) \leq k-1 + t(\phi(2,k)-1) + (k-1-t)3(\phi(2,k)-1) .$$

Now if $t < \frac{3}{5}(k-1)$, inequality (1.3.1) implies the result of the theorem and if $t \geq \frac{3}{5}(k-1)$ the result holds by inequality (1.3.2).

§1.4 Some remarks on $\phi(3,k)$.

One possible approach to obtain improved estimates on the size of $\phi(3,k)$ is to consider it as a graph theory problem as we did the case $\phi(2,k)$. The graph of a family $\mathcal{F}$ of $\phi(3,k)$ 3-sets is a generalized graph, G, where the edges of G are triangles. Given a maximal collection, $\mathcal{J}$, of $k-1$ pairwise disjoint triangles in G we call these <u>dark</u> and all others <u>light</u>. To each triangle of G we associate a distinct node of the same color and define an <u>alternating tree</u> of G as follows: An alternating tree of the graph G is a collection, $\mathcal{T}$,

of triangles of G, $\mathcal{T} = \{T_1, T_2, \ldots, T\}$, satisfying

(i) any pair of light triangles of $\mathcal{T}$ have empty intersection

(ii) any triangle in $\mathcal{I}$ having non-empty intersection with some light member of $\mathcal{T}$ belongs to $\mathcal{T}$.

(iii) there exists a spanning tree on the corresponding set of nodes such that

(a) two nodes are adjacent only if the corresponding triangles have non-empty intersection

(b) adjacent nodes are of opposite color

(c) all the endpoints correspond to light triangles of G.

The following lemma may be proved in the same way as Lemma 1.2.1:

Lemma 1.4.1 The collection of independent triangles $\mathcal{I}$ in G is maximal if and only if G contains no alternating trees.

Defining the weight of a member of $\mathcal{I}$ in a manner analagous to the weight function defined in the $\phi(2,k)$ case it would be of some advantage to prove a lemma analagous to Lemma 1.2.2. However we have not been able to prove such a lemma in this direction.

§1.5 An improved upper bound for $\phi(n,k)$.

Previously we have mentioned that Erdös and Rado [15] proved

$$\phi(n,k) \leq n!(k-1)^n \left\{ 1 - \sum_{t=1}^{n-1} \frac{t}{(t+1)!(k-1)^t} \right\} . \tag{1.5.1}$$

We now wish to establish the following theorem:

<u>Theorem 1.5.1</u> For all positive integers n and k, $k \geq 3$,

$$\phi(n,k) \leq (n+1)! \left\{ \frac{k-1+(k^2+6k-7)^{\frac{1}{2}}}{4} \right\}^n .$$

It is clear from the preceding sections that Theorem 1.5.1 holds for $n = 1$ or 2. In what follows we will assume that $n \geq 3$ and let $\mathcal{F}$ be a family of sets, each with n elements, no k members of which have pairwise the same intersection.

We shall establish the following two term recurrence inequality for $\phi(n,k)$

$$\phi(n,k) \leq \frac{k-1}{2}\{(n+1)\phi(n-1,k)+(n-1)^2\phi(n-2,k)-n(n-1)\} . \tag{1.5.2}$$

Assume for the present that (1.5.2) has been proved. Then it is easy to deduce Theorem 1.5.1 by induction. Let c be the positive root of the equation

$$2c^2 - (k-1)c - (k-1) = 0 \tag{1.5.3}$$

so that

$$c = \frac{k-1+(k^2+6k-7)^{\frac{1}{2}}}{4} .$$

By the remark made above we have for $n = 1,2$

$$\phi(n,k) < (n+1)!\ c^n .$$

Let $n \geq 3$ and assume

$$\phi(\ell,k) < (\ell+1)!\ c^\ell$$

for $\ell \leq n-1$. Then by (1.5.2) we have

$$\phi(n,k) \leq \frac{k-1}{2}\{(n+1)!c^{n-1} + (n-1)^2(n-1)!\ c^{n-2} - n(n-1)\}$$

$$< (n+1)!\ c^n \left\{ \frac{k-1}{2c} + \frac{(n-1)^2(k-1)}{2c^2 n(n+1)} \right\}$$

$$\leq (n+1)!\ c^n \left\{ \frac{(k-1)c+(k-1)}{2c^2} - \frac{3n-1}{2c^2 n(n+1)} \right\} .$$

It now follows from (1.5.3) that for $n \geq 1$

$$\phi(n,k) < (n+1)!\ c^n \quad .$$

To complete the proof of Theorem 1.5.1 we must now prove (1.5.2). We assume that $\mathcal{F}$ is maximal, that is $|\mathcal{F}| = \phi(n,k)$. This implies that there are $k-1$ sets in $\mathcal{F}$ which are pairwise disjoint. Let these sets be

$$F_i = \{\lambda \mid (i-1)n+1 \leq \lambda \leq in\} \qquad i = 1,2,\ldots,k-1 .$$

Let $\mathcal{F}^*$ be the family obtained from $\mathcal{F}$ by deleting the sets F_i , $i = 1,2,\ldots,k-1$ and for $j = 1,2,\ldots,(k-1)n$, let

$$\mathcal{A}_j = \{F \mid F \in \mathcal{F}^* , j \in F\} \quad .$$

Call a set $F \in \mathcal{F}^*$ <u>$\mathcal{F}$-disjoint</u> if F belongs to exactly one of the families $\mathcal{A}_j$. For notational convenience, for $1 \leq i \leq k-1$, $1 \leq j \leq n$, we denote the integer $(i-1)n + j$ by (i,j) .

We now prove a number of lemmas and from these deduce (1.5.2).

<u>Lemma 1.5.1</u> Let T be an $\mathcal{F}$-disjoint set in $\mathcal{A}_{(i,j)}$. Then any $\mathcal{F}$-disjoint set in $\mathcal{A}_{(i,\ell)}$ has non-empty intersection with T.

<u>Proof</u>: Let S be an $\mathcal{F}$-disjoint set in $\mathcal{A}_{(i,\ell)}$. Then if $S \cap T = \phi$, the k sets $S, T, F_1, F_2, \ldots, F_{i-1}, F_{i+1}, \ldots, F_{k-1}$ are pairwise disjoint. This is a contradiction and Lemma 1.5.1 is proved.

<u>Lemma 1.5.2</u> Let M_i be the number of $\mathcal{F}$-disjoint sets in the family $\bigcup\limits_{t=(i,1)}^{(i,n)} \mathcal{A}_t$. Then

$$M_i \le \phi(n-1,k) + (n-1)^2 \{\phi(n-2,k)-1\} - 1 . \tag{1.5.4}$$

<u>Proof</u>: If $M_i = 0$ there is nothing to prove. Hence we may suppose $M_i > 0$. This implies that for some ℓ, $1 \le \ell \le n$, there is a set $T \in \mathcal{A}_{(i,\ell)}$ which is $\mathcal{F}$-disjoint. The number of $\mathcal{F}$-disjoint sets in $\mathcal{A}_{(i,\ell)}$ is at most $|\mathcal{A}_{(i,\ell)}| \le \phi(n-1,k)-1$. If S is an $\mathcal{F}$-disjoint set in $\mathcal{A}_{(i,r)}$, $1 \le r \le n$, $r \neq \ell$, then by Lemma 1.5.1, $T \cap S \neq \phi$. There are at most $(n-1)\{\phi(n-2,k)-1\}$ such sets S in $\mathcal{A}_{(i,r)}$. Thus

$$M_i \le |\mathcal{A}_{(i,\ell)}| + \sum_{\substack{r=1 \\ r \neq \ell}}^{n} (n-1)\ \{\phi(n-2,k)-1\}$$

which implies (1.5.4).

<u>Corollary 1.5.2</u> Let M denote the number of $\mathcal{F}$-disjoint sets in $\bigcup\limits_{i=1}^{(k-1)n} \mathcal{A}_i$. Then

(1.5.5) $$M \leq (k-1)\ \{\phi(n-1,k)+(n-1)^2(\phi(n-2,k)-1)-1\}\quad .$$

Proof: This follows from (1.5.4) and the fact that

$$M = \sum_{i=1}^{k-1} M_i \quad .$$

Lemma 1.5.3 Let N be the number of sets in $\mathcal{F}^*$ which are not $\mathcal{F}$-disjoint. Then

(1.5.6) $$N \leq \frac{n(k-1)}{2}\ (\phi(n-1,k)-1) - \frac{M}{2}\quad .$$

Proof: Each set which is not $\mathcal{F}$-disjoint belongs to at least two of the families $\mathcal{A}_i$. Thus

$$\begin{aligned} N &\leq \frac{1}{2} \sum_{i=1}^{k-1} \left\{ \sum_{j=1}^{n} |\mathcal{A}_{(i,j)}| - M_i \right\} \\ &\leq \frac{1}{2} \sum_{i=1}^{k-1} \{n(\phi(n-1,k)-1) - M_i\} \\ &= \frac{n(k-1)}{2}\ (\phi(n-1,k)-1) - \frac{M}{2} \end{aligned}$$

as required.

Inequality (1.5.2) now follows from (1.5.5) and (1.5.6) and the fact that $\phi(n,k) = k-1+M+N$. This completes the proof of Theorem 1.5.1.

§1.6 A new lower bound for $\phi(n,3)$ and $\phi(n,4)$.

The main result we wish to prove in this section is

Theorem 1.6.1 For some constant c and all sufficiently large n

$$\phi(n,3) > 10^{\frac{n}{2} - c \log n} .$$

Proof: Since we shall be concerned only with the case $k = 3$ we will write $\phi(n)$ for $\phi(n,3)$. Define a function ψ as follows: $\psi(n)$ is the largest integer for which there exists a family $\mathcal{F}$ of $\psi(n)$ sets such that

(a) each set has n elements

(b) no three members of $\mathcal{F}$ have pairwise the same intersection

(c) any two members of $\mathcal{F}$ have non-empty intersection.

It is easy to see that $\psi(2) = 3$ and in fact $\{(1,2),(1,3),(2,3)\}$ is a family with the required properties. It can be shown that $\psi(3) = 10$ and that $\{(1,2,3),(1,2,4),(1,3,5),(1,4,6),(1,5,6),(2,3,6),(2,4,5),(2,5,6),(3,4,5),(3,4,6)\}$ is a family with the required properties. (See [4].) We write $\mathcal{F} \in P(n)$ if $\mathcal{F}$ satisfies (a) and (b) and $\mathcal{F} \in Q(n)$ if $\mathcal{F}$ satisfies (a), (b) and (c).

Lemma 1.6.1 $\phi(n) \geq 2\psi(n)$.

Proof: Let $\mathcal{F}_1 \in Q(n)$ and $\mathcal{F}_2 \in Q(n)$ and suppose each set in $\mathcal{F}_1$ is disjoint from each set in $\mathcal{F}_2$. Then if $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ we have $\mathcal{F} \in P(n)$. This prove Lemma 1.6.1.

Lemma 1.6.2 $\psi(a+b) \geq \psi(a)\phi(b)$.

Proof: Let $\mathcal{F}_1 \varepsilon Q(a)$ and $\mathcal{F}_2 \varepsilon P(b)$. Let

$$\mathcal{F} = \{F \mid F = F_1 \cup F_2,\ F_1 \varepsilon \mathcal{F}_1,\ F_2 \varepsilon \mathcal{F}_2\} .$$

We need to show that $\mathcal{F} \varepsilon Q(a+b)$. That $\mathcal{F} \varepsilon P(a+b)$ can be proved by the argument used in [1] to prove $\phi(a+b) \geq \phi(a)\phi(b)$. Hence to complete the proof we need only to show that any two sets in $\mathcal{F}$ have non-empty intersection. Let G_1 and G_2 be two distinct members of $\mathcal{F}$. We have $G_i = F_{i,1} \cup F_{i,2}$ where $F_{i.1} \varepsilon \mathcal{F}_1$ and $F_{i,2} \varepsilon \mathcal{F}_2$. Since $\mathcal{F}_1 \varepsilon Q(a)$ we must have $F_{1,1} \cap F_{2,1} \neq \emptyset$ and hence $G_1 \cap G_2 \neq \emptyset$. Thus $\mathcal{F} \varepsilon Q(a+b)$.

Lemma 1.6.3 $\quad \psi(ab) \geq \psi(a)\psi(b)^a$.

Proof: Let $\mathcal{F} \varepsilon Q(a)$ and let $\cup\mathcal{F} = \{1,2,\ldots,\ell\}$. For $j = 1,2,\ldots,\ell$ let $\mathcal{F}_j \varepsilon Q(b)$. We assume that all families are maximal and disjoint (in the sense that $F \varepsilon \mathcal{F}_i$, $G \varepsilon \mathcal{F}_j$ implies $F \cap G = \emptyset$). Let $F \varepsilon \mathcal{F}$. We have $F = \{i_1, i_2, \ldots, i_a\}$, say. From each of the families $\mathcal{F}_{i_1}, \mathcal{F}_{i_2}, \ldots, \mathcal{F}_{i_a}$ select a set and let F^* be the union of the sets selected. We shall say that F generates F^*. Let $\mathcal{F}^*$ be the family of sets generated this way. It is clear that each $F \varepsilon \mathcal{F}$ generates $\psi(b)^a$ sets and hence $|\mathcal{F}^*| = \psi(a)\psi(b)^a$. Moreover each member of $\mathcal{F}^*$ has ab elements. To complete the proof we must show that $\mathcal{F}^* \varepsilon Q(a,b)$.

First we show that $\mathcal{F}^* \varepsilon P(ab)$. Let F_1^*, F_2^* and F_3^* be distinct members of $\mathcal{F}^*$ and let F_1, F_2 and F_3 be the sets in $\mathcal{F}$ which generate F_1^*, F_2^* and F_3^* respectively. Let $F_1 \cap F_2 \cap F_3 = K$ and let $K_{ij} = (F_i \cap F_j) - K$, $1 \leq i < j \leq 3$, so that the sets K, K_{12}, K_{13}

and K_{23} are pairwise disjoint.

Suppose $K_{12} \neq \emptyset$ and let $i \in K_{12}$. Then there exists sets F_i^1 and F_i^2 in $\mathcal{F}_i$ such that $F_i^1 \subset F_1^*$ and $F_i^2 \subset F_2^*$ but there exists no set in $\mathcal{F}_i$ which is a subset of F_3^*. Since $F_i^1 \cap F_i^2 \neq \emptyset$, F_1^*, F_2^*, and F_3^* do not have pairwise the same intersection. Hence we may assume $K_{12} = \emptyset$. Similarly $K_{13} = K_{23} = \emptyset$. This means that F_1, F_2 and F_3 have pairwise the same intersection, but $\mathcal{F} \in P(a)$, hence $F_1 = F_2 = F_3$, i.e. F_1^*, F_2^* and F_3^* have a common generator, say $F = \{i_1, i_2, \ldots, i_a\}$. Then

$$F_r^* = \bigcup_{j=1}^{a} F_{i_j}^{(r)} \qquad \text{for} \quad r = 1,2,3$$

where $F_{i_j}^{(r)} \in \mathcal{F}_{i_j}$. Since F_1^*, F_2^* and F_3^* are distinct sets, there is a least integer m, $1 \leq m \leq a$, such that $F_{i_m}^{(1)}$, $F_{i_m}^{(2)}$, and $F_{i_m}^{(3)}$ are not all equal and, since $\mathcal{F}_{i_m} \in P(a)$, do not have pairwise the same intersection. It follows that $\mathcal{F}^* \in P(ab)$.

Now we prove that $\mathcal{F}^* \in Q(ab)$. Let F^* and G^* be distinct members of $\mathcal{F}^*$ and let F and G be the generating sets. Since $\mathcal{F} \in Q(a)$, $F \cap G \neq \emptyset$. Thus, if $i \in F \cap G$, there exists sets $F', G' \in \mathcal{F}_i$ such that $F' \subset F^*$ and $G' \subset G^*$. Moreover, since $\mathcal{F}_i \in Q(b)$, $F' \cap G' \neq \emptyset$ and hence $F^* \cap G^* \neq \emptyset$. Thus $\mathcal{F}^* \in Q(ab)$. This completes the proof of the lemma.

We now complete the proof of Theorem 1.6.1. From Lemma 1.6.3 and the fact that $\psi(3) = 10$ we have for $k \geq 2$

$$\psi(3^k) \geq \psi(3)\psi(3^{k-1})^3 > 10\psi(3^{k-1})^3 \tag{1.6.1}$$

and it is a simple matter to verify that (1.6.1) implies

$$\psi(3^k) \geq 10^{\frac{3^k-1}{2}} \tag{1.6.2}$$

Now let n be an integer and write n in base 3. Then by (1.6.2) and Lemma 1.6.2 we have (using $\psi(2) = 3$, $\phi(2) = 6$)

$$\psi(n) > 10^{\frac{n}{2} - c \log n} \tag{1.6.3}$$

for some constant c and all n sufficiently large. Theorem 1.6.1 now follows from (1.6.3) and Lemma 1.6.1.

The recurrence inequalities obtained in Lemmas 1.6.1 and 1.6.3 provide substantially better lower bounds for $\phi(n,3)$ even for small values of n. For example if we take $a = b = 2$ in Lemma 1.6.3 we have $\psi(4) \geq 27$ and hence by Lemma 1.6.1, $\phi(4,3) \geq 54$. Note that (1.1.4) yields only $\phi(4,3) \geq 16$. If we take $a = 2, b = 3$ we have $\psi(6) \geq 300$ and consequently $\phi(6,3) \geq 600$, whereas (1.1.4) yields only $\phi(6,3) \geq 64$.

If we define $\psi(n,k)$ to be the largest integer for which there exists a family $\mathcal{F}$ of $\psi(n,k)$ sets such that

(a) each set has n elements

(b) no k member of $\mathcal{F}$ have pairwise the same intersection

(c) among any $[\frac{k}{2}] + 1$ members of $\mathcal{F}$ there are at least two sets which have non-empty intersection.

then it is not difficult to prove

Lemma 1.6.4 $\phi(n,k) \geq 2\psi(n,k)$ if k is odd

$\phi(n,k) \geq \psi(n,k)+\psi(n,k-1)$ if k is even.

The following family of sets shows that $\psi(3,4) \geq 26$:

$$F = \{(1,2,3),(1,2,4),(1,2,5),(1,3,6),(1,3,7),(1,4,8),(1,4,5),(1,5,6),$$
$$(1,6,7),(1,7,8),(2,4,6),(2,4,7),(2,5,6),(2,5,7),(2,3,8),(2,6,8),$$
$$(2,7,8),(3,5,7),(3,6,8),(3,5,6),(3,4,7),(3,4,8),(3,4,5),(4,5,8),$$
$$(4,5,6),(5,7,8)\} .$$

As we remarked previously, $\psi(3,3) = 10$, hence we have

Corollary 1.6.4 $\phi(3,4) \geq 36$.

In [1] Abbott has shown that

$$\phi(a+b,k) \geq \phi(a,k)\phi(b,k)$$

and this together with $\phi(3,4) \geq 36$ implies

$$\phi(n,4) \geq \begin{cases} 36^{\frac{n}{3}} & \text{if } n = 3t \\ 3\cdot 36^{\frac{n-1}{3}} & \text{if } n = 3t+1 \\ 10\cdot 36^{\frac{n-2}{3}} & \text{if } n = 3t+2 \end{cases} ,$$

which is a substantial improvement on (1.1.4) when $k = 4$. However we have not been able to find any useful generalization of Lemma 1.6.3 to the case $k > 3$ which would lead to further improvements of this lower bound.

§1.7 An application to a problem in number theory.

In [9], P. Erdös considered the following problem in number theory: Given positive integers n and k, $n \geq k \geq 3$, what is the largest integer $f(n,k)$ for which there exists a set S of $f(n,k)$ integers satisfying

(i) $S \subseteq \{1,2,\ldots,n\}$

(ii) no k members of S have pairwise the same greatest common divisor.

The problem of determining $f(n,k)$ appears to be difficult and the upper and lower bounds which have been obtained are very far apart. In [1], H.L. Abbott proved the following: for every $\varepsilon > 0$ and every fixed m and k,

$$f(n,k) \geq \phi(m,k)^{\frac{\log n}{(1+\varepsilon)m \log\log n}}$$

provided $n \geq n_o(m,k,\varepsilon)$. In particular, by the results of the preceding section we have

$$f(n,3) > 10^{\frac{\log n}{2(1+\varepsilon)\log\log n}}$$

for $n \geq n_o(\varepsilon)$, and in general

$$f(n,k) > n^{\frac{\log (k-1)}{(1+\varepsilon)\log\log n}}$$

for $n \geq n_o(\varepsilon)$. On the other hand in [9] Erdös has shown that for each fixed integer $k \geq 3$ and each $\varepsilon > 0$

$$f(n,k) < n^{\frac{3}{4}+\varepsilon} \tag{1.7.1}$$

provided n is sufficiently large. In the case where $k \to \infty$ with n it is known that for $0 < \alpha < 1$,

$$f(n,[n^{\alpha}]) = c_{\alpha}n(1+o(1))$$

where c_{α} is a constant depending only on α (see [9]), and that for $\alpha > 0$, $\varepsilon > 0$

$$(1.7.2) \qquad n^{\frac{\alpha}{\alpha+1} - \varepsilon} < f(n,[\log^{\alpha} n]) < n^{\frac{2\alpha+3}{2\alpha+4} + \varepsilon}$$

provided n is sufficiently large (see [3]).

In this section we wish to prove the following theorem:

Theorem 1.7.1 For each fixed integer $k \geq 3$ and each $\varepsilon > 0$

$$(1.7.3) \qquad f(n,k) < n^{\frac{1}{2} + \varepsilon}$$

and for each $\alpha > 0$, $\varepsilon > 0$

$$(1.7.4) \qquad f(n,[\log^{\alpha} n]) < n^{\frac{\alpha+1}{\alpha+2} + \varepsilon}$$

provided, in each case n is sufficiently large.

It is clear that (1.7.3) and (1.7.4) give better upper bounds than those given by (1.7.1) and (1.7.2). We prove first the following lemma, the proof of which is a modification of the argument used in [9].

Lemma 1.7.1 Let $f^*(n,k)$ be the largest number of square free integers that one can select from $\{1,2,\ldots,n\}$ no k with pairwise the same

greatest common divisor. Then for fixed $k \geq 3$ and $\varepsilon > 0$

$$(1.7.5) \qquad f^*(n,k) < n^{\frac{1}{2}+\varepsilon}$$

and for each $\alpha > 0$, $\varepsilon > 0$

$$(1.7.6) \qquad f^*(n,[\log^\alpha n]) < n^{\frac{\alpha+1}{\alpha+2}+\varepsilon}$$

provided n is sufficiently large.

Proof: Let $\ell = [n^{\frac{1}{2}+\varepsilon}]$ and let $S \subseteq \{1,2,\ldots,n\}$ be a set of ℓ square free integers, no k of which have pairwise the same greatest common divisor. Let $S = S_1 \cup S_2$ where

$$S_1 = \{a \mid a \in S, \omega(a) \geq u\}$$

and

$$S_2 = \{a \mid a \in S, \omega(a) < u\} \quad .$$

Here $\omega(a)$ denotes the number of prime divisors of a and u is an integer to be specified later. Let

$$B = \{b \mid b \leq n, \text{ b square free, } \omega(b) = u\} \ .$$

Then every number in S_1 is a multiple of some number in B . Hence we have

$$|S_1| \leq \sum_{b\varepsilon B} [\tfrac{n}{b}] \leq n \sum_{b\varepsilon B} \frac{1}{b} \leq \frac{n}{u!} \left(\sum_{p\leq n} \frac{1}{p} \right)^u$$

$$= \frac{n}{u!} (\log \log n + O(1))^u \quad .$$

Choose $u = [\log n/2 \log \log n]$. Then a straight forward calculation

shows that

$$|S_1| = o(n^{\frac{1}{2}+\varepsilon}) = o(\ell) .$$

Hence, for n sufficiently large,

$$|S_2| \geq \frac{\ell}{2} . \tag{1.7.7}$$

The theorem of Erdös and Rado [15] discussed earlier asserts that any family of more than $u!(k-1)^u$ sets, each set with u or fewer elements, contains k sets with pairwise the same intersection.

Therefore we have

$$|S_2| \leq u!\ (k-1)^u , \tag{1.7.8}$$

and by (1.7.7)

$$\frac{\ell}{2} \leq u!\ (k-1)^u . \tag{1.7.9}$$

However, a routine calculation shows that with $\ell = [n^{\frac{1}{2}+\varepsilon}]$ and $u = [\log n/2 \log \log n]$, (1.7.9) is false. This completes the proof of (1.7.5). The proof of (1.7.6) is similar to the proof of (1.7.5), except that one takes $\ell = [n^{\frac{\alpha+1}{\alpha+2}+\varepsilon}]$ and $u = [\log n/(2+\alpha) \log \log n]$. We omit the details. This completes the proof of the lemma.

<u>Proof of Theorem 1.7.1</u> To prove (1.7.3) let $S \subseteq \{1,2,\ldots,n\}$ be a set of $f(n,k)$ integers no k of which have pairwise the same greatest common divisor. Let $S = \bigcup_{i=1}^{m} S_i$ where $m = [\sqrt{n}]$ and S_i consists of those integers in S which are divisible by i^2 but by no larger square.

Given $\varepsilon > 0$, we now choose n large enough such that if we take n to be $n^{2\varepsilon}$ in (1.7.5) the result of the lemma still holds. Then by Lemma 1.7.1, we have

$$f(n,k) = |S| = \sum_{i=1}^{m} |S_i| \le \sum_{i=1}^{m} f^*\left(\left[\frac{n}{i^2}\right],k\right)$$

$$= \sum_{i=1}^{[n^{\frac{1}{2}-\varepsilon}]} f^*\left(\left[\frac{n}{i^2}\right],k\right) + \sum_{i=[n^{\frac{1}{2}-\varepsilon}]+1}^{m} f^*\left(\left[\frac{n}{i^2}\right],k\right)$$

$$< \sum_{i=1}^{[n^{\frac{1}{2}-\varepsilon}]} \left(\frac{n}{i^2}\right)^{\frac{1}{2}+\varepsilon} + \sum_{i=[n^{\frac{1}{2}-\varepsilon}]+1}^{m} \frac{n}{i^2} < n^{\frac{1}{2}+2\varepsilon} .$$

This implies (1.7.3). The proof of (1.7.4) is similar and we omit the details.

CHAPTER II

A PROBLEM OF SCHUR AND ITS GENERALIZATIONS

§2.1 A problem of Schur

A set S of integers is said to be <u>sum-free</u> if $a,b \in S$ implies $a+b \notin S$ (a and b not necessarily distinct).

A well known theorem of I. Schur [25] states that if the integers $1,2,\ldots,[n!e]$ are partitioned in any manner into n classes, then at least one of the classes is not sum-free. Accordingly we define $f(n)$ to be the largest positive integer such that the integers $1,2,\ldots,f(n)$ can be partitioned in some manner into n sum-free classes.

It is easy to verify that $f(1) = 1$, $f(2) = 4$ and $f(3) = 13$. In 1961 L.D. Baumert [7] with the aid of a high speed computer, showed that $f(4) = 44$. One of the many such partitionings of the integers $1,2,\ldots,44$ that he found is as follows:

$$C_1 = \{1,3,5,15,17,19,26,28,40,42,44\}$$

$$C_2 = \{2,7,8,18,21,24,27,33,37,38,43\}$$

$$C_3 = \{4,6,13,20,22,23,25,30,32,39,41\}$$

$$C_4 = \{9,10,11,12,14,16,29,31,34,35,36\} .$$

The value of $f(n)$ for $n > 4$ is not known and it appears very difficult to determine $f(n)$, even for $n = 5$.

In [25] Schur proved that

$$f(n+1) \geq 3f(n) + 1 \tag{2.1.1}$$

and as a result of this

$$f(n) \geq \frac{3^n - 1}{2} \quad . \tag{2.1.2}$$

Defining $g(m)$ to be the smallest number of sum-free classes into which the integers $1,2,\ldots,m$ can be partitioned, H.L. Abbott and L. Moser [6] showed that for all positive integers p and q

$$f(pq+g(pf(q))) \geq (2f(q)+1)^p - 1 \quad . \tag{2.1.3}$$

From this they deduce that for some absolute constant c and all n sufficiently large

$$f(n) > 89^{\frac{n}{4} - c \log n} \quad , \tag{2.1.4}$$

which improves Schur's lower bound. On the other hand Schur's theorem states

$$f(n) \leq [n!e] - 1 \; . \tag{2.1.5}$$

In this section we obtain a lower bound for $f(n)$ which is better than that given by (2.1.4). However, instead of studying $f(n)$ directly we consider the following more general problem.

Let $f_k(n)$ be defined as follows: $f_k(n)$ is the largest positive integer such that the integers $1,2,\ldots,f_k(n)$ can be partitioned into n classes, no class containing a solution to the following system, (S) , of $\binom{k-1}{2}$ equations in $\binom{k}{2}$ unknowns:

$$x_{i,j} + x_{j,j+1} = x_{i,j+1} \qquad 1 \leq i < j \leq k-1 \quad .$$

We will call such classes (S)-free. It is easy to see that $f_3(n) = f(n)$. That $f_k(n)$ exists for $k > 3$ follows from the results of R. Rado [21]. Now define $g_k(m)$ as follows: If $f_k(n-1) < m \leq f_k(n)$, then $g_k(m) = n$; i.e. $g_k(m)$ is the smallest number of (S)-free classes into which the integers $1,2,\ldots,m$ can be partitioned. E.R. Williams [28] has shown for all positive integers p and q that

(2.1.6) $$f_k(pq+g_k(pf_k(q))) \geq (2f_k(q)+1)^p - 1 \quad .$$

This was proven analogously to the work of Abbott and Moser and reduces, in the case $k = 3$, to their result (2.1.3).

We now prove by a new method the following theorem which improves the results (2.1.3) and (2.1.6):

<u>Theorem 2.1.1</u> For all positive integers n and m

$$f_k(n+m) \geq (2f_k(m)+1)f_k(n) + f_k(m) \quad .$$

<u>Proof</u>: Partition the integers $1,2,\ldots,f_k(m)$ into m (S)-free classes $C_1,C_2,\ldots,C_m$. Let

$$A = \{a(2f_k(m)+1)+j \mid a = 0,1,\ldots,f_k(n),\ j = 1,2,\ldots,f_k(m)\} \quad .$$

Partition A into m classes $C'_1,C'_2,\ldots,C'_m$ by placing $a(2f_k(m)+1)+j$ in C'_i if $j \in C_i$. Partition the integers $1,2,\ldots,f_k(n)$ into n (S)-free classes $D_1,D_2,\ldots,D_n$. Let

$$B = \{a(2f_k(m)+1)-j \mid a = 1,2,\ldots,f_k(n),\ j = 0,1,\ldots,f_k(m)\} \quad .$$

Partition B into n classes $C'_{m+1},C'_{m+2},\ldots,C'_{m+n}$ by placing $a(2f_k(m)+1)-j$

in C'_{m+i} if $a \varepsilon D_i$. It is easy to see that $A \cap B = \emptyset$ and $A \cup B = \{1,2,\ldots,(2_k(m)+1)f_k(n) + f_k(m)\}$ and thus we have partitioned the integers $1,2,\ldots,(2f_k(m)+1)f_k(n) + f_k(m)$ into $m+n$ classes $C_1',C'_2,\ldots,C'_{m+n}$ and it remains only to show that each of these classes is (S)-free.

Consider first the classes $C_1',C_2',\ldots,C_m'$. If one of these classes, say C_r' , contains a solution to (S) it is of the form

$$(2.1.7) \qquad a_{i,j}(2f_k(m)+1) + b_{i,j} + a_{j,j+1}(2f_k(m)+1) + b_{j,j+1}$$
$$= a_{i,j+1}(2f_k(m)+1) + b_{i,j+1}$$

where $1 \leq i < j \leq k-1$, $1 \leq b_{s,t} \leq f_k(m)$, and $b_{s,t} \varepsilon C_r$. Equation (2.1.7) implies

$$(a_{i,j}+a_{j,j+1}-a_{i,j+1})(2f_k(m)+1) = b_{i,j+1} - b_{i,j} - b_{j,j+1} ,$$

and hence $b_{i,j+1} - b_{i,j} - b_{j,j+1} \equiv 0(\text{mod } 2f_k(m)+1)$. But then we must have $b_{i,j} + b_{j,j+1} = b_{i,j+1}$ for all $1 \leq i < j \leq k-1$, since $1 \leq b_{s,t} \leq f_k(m)$ for all possible choices of s and t . This is a contradiction of the fact C_r is (S)-free.

Now consider the classes $C'_{m+1},C'_{m+2},\ldots,C'_{m+n}$. Suppose some class, say $C'_{m+\ell}$, $1 \leq \ell \leq n$, contains a solution to (S) . Then we have

$$(2.1.8) \qquad a_{i,j}(2f_k(m)+1) - b_{i,j} + a_{j,j+1}(2f_k(m)+1) - b_{j,j+1}$$
$$= a_{i,j+1}(2f(m)+1) - b_{i,j+1}$$

where $1 \leq i < j \leq k-1$ and $0 \leq b_{s,t} \leq f_k(m)$ and where $a_{s,t}$ belongs to D_ℓ for all possible choices of s and t. As in the previous case we must have that $b_{i,j} + b_{j,j+1} = b_{i,j+1}$ for all $1 \leq i < j \leq k-1$ and consequently equation (2.1.8) implies $a_{i,j} + a_{j,j+1} = a_{i,j+1}$ for all $1 \leq i < j \leq k-1$. This is a contradition by the construction of the class D_ℓ.

Corollary 2.1.1 For all positive integers m and n

$$f(n+m) \geq (2f(m)+1)f(n) + f(m) \quad .$$

Proof: Let $k = 3$ in Theorem 2.1.1.

Corollary 2.1.2 For $n \geq 4$, and for some absolute constant c,

$$f(n) \geq c\ 89^{\frac{n}{4}} \quad .$$

Proof: By Corollary 2.1.1 we have $f(n+4) \geq 89f(n) + 44$, and this implies the result with $c = \frac{44}{89}$.

It is clear that the lower bound for $f(n)$ given by Corollary 2.1.2 is better than that given by (2.1.4).

Corollary 2.1.3 For $n \geq 1$ and for some constant c_k, dependent only on k,

$$f_k(n) \geq c_k(2k-3)^n \quad .$$

Proof: Since $f_k(1) = k-2$, the result follows from Theorem 2.1.1 with $c_k = \frac{k-2}{2k-3}$.

§2.2 Some applications to Ramsey's Theorem

In 1930, F.P. Ramsey [22] published a combinatorial theorem which may be formulated as follows:

Ramsey's Theorem Let n, k and r be positive integers with $k \geq r$. Then there exists a least positive integer $R_n(k,r)$ such that if $s \geq R_n(k,r)$, S is a set of s elements, and the collection of subsets of S with r elements is partitioned in an arbitrary manner into n classes, then there is some subset K of S with k elements such that the subsets of K with r elements all belong to the same class.

In this section we shall be concerned only with the case $r = 2$. We may then reformulate Ramsey's Theorem in this special case as follows: If G is a complete graph on $R \geq R_n(k,2)$ vertices and if each edge of G is colored in any one of n colors, then there results a complete subgraph of G on k vertices, all of whose edges have the same color, i.e. a complete monochromatic k-gon.

Many studies have been done on $R_n(k,2)$ but the problem of evaluating this function appears very difficult even for small values of n and k. P. Erdös [10] and H.L. Abbott [2] have shown that

$$R_2(k,2) > c\,k\,2^{\frac{k}{2}} \tag{2.2.1}$$

from some constant c. The argument used by Erdös to prove (2.2.1) can be used to prove

$$\binom{R_n(k,2)}{k} \geq n^{\binom{k}{2}-1} . \tag{2.2.2}$$

This gives a lower bound of approximately $kn^{\frac{k}{2}}$. On the other hand R.E. Greenwood and A.M. Gleason [17] have shown

$$R_n(k,2) \le \frac{(nk-n)!}{((k-1)!)^n}$$

and in the particular case $k = 3$ that

$$R_n(3,2) \le [n!e] + 1 . \tag{2.2.3}$$

In this section we shall be concerned with estimating a lower bound for $R_n(k,2)$ for some small values of k . In this direction the best previous results are those of Guy R. Giraud [16] . Giraud has shown for $n \ge 2$

$$R_n(4,2) \ge \frac{33}{2}\, 5^{n-2} + \frac{3}{2}$$

and

$$R_n(5,2) \ge \frac{73}{2}\, 7^{n-2} + \frac{3}{2} .$$

Here we shall improve these results.

Let $f_k(n)$ and the system (S) be defined as in Section 2.1. Partition the integers $1,2,\ldots,f_k(n)$ into n (S)-free classes $C_1,C_2,\ldots,C_n$. Let G be a graph with vertices $P_0,P_1,\ldots,P_{f_k(n)}$. Color the edge (P_i,P_j) color c_r if $|i-j| \in C_r$. Suppose that $P_{i_1},P_{i_2},\ldots,P_{i_k}$, where $i_1 > i_2 > \ldots > i_k$, are the vertices of a monochromatic k-gon of color c_r . Then $i_t - i_s \in C_r$ for $1 \le t < s \le k$. But then

$$(i_t - i_s) + (i_s - i_{s+1}) = (i_t - i_{s+1}) \qquad 1 \le t < s \le k-1$$

is a solution to the system (S) in C_r , a contradiction. Therefore we have

$$R_n(k,2) \ge f_k(n) + 2 \quad . \tag{2.2.3'}$$

Equation (2.2.3') together with the result of Greenwood and Gleason (2.2.2) imply Schur's result (2.1.5).

Theorem 2.2.1 For $n \ge 1$, $k \ge 2$ and for some constant c_k , dependent only on k

$$R_n(k,2) \ge c_k (2k-3)^n \ .$$

Proof: This is an immediate consequence of equation (2.2.3) and Corollary 2.1.3.

Theorem 2.2.1 as opposed to the inequalities (2.2.1) and (2.2.2) is effective when k is small and n is large. The theorem could perhaps be improved substantially if some new estimates for $f_k(n)$ could be found for $n > 1$. Although one might conjecture that $f_k(n)$ grows like $R_n(k,2)$ we cannot obtain any useful estimates of even $f_k(2)$. However in certain special cases we can improve the lower bound given by Theorem 2.2.1.

Theorem 2.2.2 For $n \ge 4$ and for some constant c

$$R_n(3,2) \ge c\, 89^{\frac{n}{4}} \ .$$

Proof: This is an immediate consequence of (2.2.3) and Corollary 2.1.2.

Theorem 2.2.3 For $n \geq 2$ and some constant c

$$R_n(4,2) \geq c\ 33^{\frac{n}{2}} .$$

Proof: Partition the integers $1,2,\ldots,16$ into the following sets:

$$C_1 = \{1,2,4,8,9,13,15,16\}$$

$$C_2 = \{3,5,6,7,10,11,12,14\} ,$$

where C_1 consists of the quadratic residues of 17 and C_2 the non residues.

From this partitioning it follows that $f_4(2) \geq 16$, since it is a routine matter to verify that C_1 and C_2 are (S)-free.

The result now follows from (2.2.3) and Theorem 2.1.1.

In a similar manner it can be shown that $f_5(2) \geq 37$ and consequently $R_n(5,2) > c\ 75^{\frac{n}{2}}$ for some constant c. However in [2] Abbott has shown for integers a and $b \geq 2$

(2.2.4) $$R_n(ab-a-b+2,2) \geq (R_n(a,2)-1)(R_n(b,2)-1) + 1 .$$

Taking $a = b = 3$ in (2.2.4) we have

(2.2.5) $$R_n(5,2) \geq (R_n(3,2)-1)^2 + 1$$

and in view of Theorem 2.2.2 we have

(2.2.6) $$R_n(5,2) > c\ 89^{\frac{n}{2}}$$

for some constant c .

§2.3 A generalization of Schur's problem

As was observed by R. Rado [21] the problem of Schur is a special case of a more general problem. Consider the following equation in ℓ unknowns $x_1, x_2, \ldots, x_\ell$:

$$\sum_{i=1}^{\ell} a_i x_i = 0 , \tag{2.3.1}$$

where $a_1, a_2, \ldots, a_\ell$ are non-zero integers. Rado called equation (2.3.1) n-fold regular if there exists a non-negative integer $f(n)$, which we take to be minimal, such that if the integers $1, 2, \ldots, f(n)+1$ are partitioned in any manner into n classes, then at least one of the classes contains a solution to (2.3.1). Equation (2.3.1) is said to be regular if it is n-fold regular for every positive integer n .

One of the main results which Rado establishes is the following criterion giving necessary and sufficient conditions for an equation to be regular: Equation (2.3.1) is regular if and only if some subset of the coefficients has zero sum. Thus the equation $x_1 + x_2 - x_3 = 0$ is regular and it is easy to see that the problem of Schur consists of finding bounds for $f(n)$ for the equation $x_1 + x_2 - x_3 = 0$. The problem of estimating lower bounds for $f(n)$ for a number of regular equations was considered by H. Salié [23].

Write (2.3.1) in the form

$$\sum_{i=1}^{t} a_i x_i = \sum_{i=t+1}^{\ell} a_i x_i \tag{2.3.2}$$

where $a_1, a_2, \ldots, a_\ell$ are positive integers. Suppose (2.3.2) is regular. Henceforth we assume that

$$A = \sum_{i=1}^{t} a_i > \sum_{i=t+1}^{\ell} a_i = B ,$$

since otherwise $f(n) = 0$ for all n.

<u>Theorem 2.3.1</u> Let m be a positive integer. Let M and N be integers satisfying

$$(A-1)f(m) \leq M < N \tag{2.3.3}$$

and

$$Af(m)+1 \leq N \leq \left\{\frac{A}{A-1}(M+1)\right\} \tag{2.3.4}$$

where

$$\{x\} = \begin{cases} [x] & \text{if } x \text{ is not an integer} \\ x-1 & \text{if } x \text{ is an integer .} \end{cases}$$

Let $h(M,N)$ be the least number of sets into which the integers $1,2,\ldots,M$ can be partitioned, no set containing a solution of any of the equations

$$\sum_{i=1}^{t} a_i x_i = \sum_{i=t+1}^{\ell} a_i x_i + \mu N \tag{2.3.5}$$

where $\mu = -B+1, -B+2, \ldots, A-1$ if $N < \frac{A}{A-1} M$ and $\mu = -B+1, -B+2, \ldots, A-2$ if $\frac{A}{A-1} M \leq N \leq \left\{\frac{A}{A-1}(M+1)\right\}$. Let $h(m) = \min h(M,N)$ where the minimum is taken over all pairs M,N satisfying (2.3.3) and (2.3.4). Then for all positive integers n

$$f(n+h(m)) \geq N_1 f(n) + M_1$$

where N_1 and M_1 satisfy (2.3.3) and (2.3.4) and $h(M_1,N_1) = h(m)$.

Proof: Partition the integers $1,2,\ldots,M_1$ into $h(m)$ classes $C_1,C_2,\ldots,C_{h(m)}$ satisfying the conditions given in defining $h(m)$. Let $A' = \{bN_1+c \mid b = 0,1,\ldots,f(n),\ c = 1,2,\ldots,M_1\}$. Partition A' into $h(m)$ classes $C'_1,C'_2,\ldots,C'_{h(m)}$ by placing bN_1+c in C'_i if $c \in C_i$. Partition the integers $1,2,\ldots,f(n)$ into n classes $D_1,D_2,\ldots,D_n$ none of which contain a solution to the given equation. Let $B' = \{bN_1-c \mid b = 1,2,\ldots,f(n),\ c = 0,1,\ldots,N_1-M_1-1\}$. Partition B' into n classes $C'_{h(m)+1}, C'_{h(m)+2}, \ldots, C'_{h(m)+n}$ by placing bN_1-c in class $C'_{h(m)+i}$ if $b \in D_i$. It is easy to see that $A' \cap B' = \emptyset$ and $A' \cup B' = \{1,2,\ldots,N_1 f(n)+M_1\}$ and thus we have partitioned the integers $1,2,\ldots,N_1 f(n)+M_1$ into $h(m)+n$ classes $C'_1,C'_2,\ldots,C'_{h(m)+n}$ and it remains only to show that none of these classes contain a solution to equation (2.3.2).

Consider first the classes $C'_1,C'_2,\ldots,C'_{h(m)}$. If any one of these classes contains a solution to equation (2.3.2) it is of the form

$$\sum_{i=1}^{t} a_i(b_iN_1+c_i) = \sum_{i=t+1}^{\ell} a_i(b_iN_1+c_i)$$

where $0 \leq b_i \leq f(n)$ and $1 \leq c_i \leq M_1$. Hence we must have

$$\sum_{i=1}^{t} a_ic_i \equiv \sum_{i=t+1}^{\ell} a_ic_i \pmod{N_1} .$$

But $0 < \sum_{i=1}^{t} a_ic_i \leq AM_1$ and $0 < \sum_{i=t+1}^{\ell} a_ic_i \leq BM_1$. Therefore if

$M_1 \leq N_1 < \frac{A}{A-1} M_1$, we have $AM_1 \leq AN_1$ and $BM_1 \leq BN_1$. Then by the definition of $h(m)$ we have a contradiction. On the other hand, if $\frac{A}{A-1} M_1 \leq N_1 \leq \{\frac{A}{A-1}(M_1+1)\}$ then $AM_1 \leq (A-1)N_1$ and $BM_1 < BN_1$ and again by the definition of $h(m)$ we have a contradiction. Therefore none of the classes $C_1', C_2', \ldots, C_{h(m)}'$ contain a solution to equation (2.3.2).

Now consider the classes $C_{h(m)+1}', C_{h(m)+2}', \ldots, C_{h(m)+n}'$. If any one of these classes contains a solution to equation (2.3.2) it is of the form

$$\sum_{i=1}^{t} a_i(b_iN_1-c_i) = \sum_{i=t+1}^{\ell} a_i(b_iN_1-c_i)$$

where $1 \leq b_i \leq f(n)$ and $0 \leq c_i \leq N_1-M_1-1$. By construction we must have either

$$\sum_{i=1}^{t} a_ib_iN_1 \leq \sum_{i=t+1}^{\ell} a_ib_iN_1+N_1$$

or

$$\sum_{i=1}^{t} a_ib_iN_1+N_1 \leq \sum_{i=t+1}^{\ell} a_ib_iN_1 \quad .$$

In the first case we must have that

$$A(N_1-M_1-1) \geq \sum_{i=1}^{t} a_ic_i \geq \sum_{i=t+1}^{\ell} a_ic_i+N_1 \geq N_1$$

which is false if $N_1 \leq \{\frac{A}{A-1}(M+1)\}$. In the second case we must have that

$$N_1 \leq N_1 + \sum_{i=1}^{t} a_ic_i \leq \sum_{i=t+1}^{\ell} a_ic_i \leq B(N_1-M_1-1) < A(N_1-M_1-1)$$

which again is false if $N \leq \{\frac{A}{A-1}(M+1)\}$. Therefore none of the classes $C'_{h(m)+1}, C'_{h(m)+2}, \ldots, C'_{h(m)+n}$ contain a solution to equation (2.3.2) and the proof of the theorem is complete.

It is now easy to see that Corollary 2.1.1 follows from Theorem 2.3.1. In this case we are considering the equation $x_1 + x_2 = x_3$ and we may choose $M_1 = f(m)$, $N_1 = 2f(m)+1$ and thus $h(m) = m$. In fact, more generally, Theorem 2.1.1 can be deduced as a corollary to Theorem 2.3.1.

Consider the regular equation

$$(2.3.6) \qquad 2x_1 + x_2 = 2x_3 .$$

H. Salié [23] proved that for equation (2.3.6) $f(n) \geq 2^n - 1$ and H.L. Abbott [2] improved this result to

$$(2.3.7) \qquad f(n) > 40^{\frac{n}{5} - c \log n}$$

for some constant c and n sufficiently large. Applying Theorem 2.3.1 to equation (2.3.6) with $m = 2$, $M_1 = 9$ and $N_1 = 12$ we have that $h(2) = 3$ as may be seen by the following partitioning of the integers $1,2,\ldots,9$:

$$C_1 = \{1,6,7\}, \quad C_2 = \{2,5,8\}, \quad C_3 = \{3,4,9\} .$$

Therefore Theorem 2.3.1 implies that for equation (2.3.6)

$$(2.3.8) \qquad f(n+3) \geq 12f(n) + 9$$

and consequently

$$(2.3.9) \qquad f(n) > c\, 12^{\frac{n}{3}}$$

for some constant c which improves (2.3.7) considerably.

Salié [23] also proved that $f(n) \geq 2^n - 1$ for the regular equation $x_1 + x_2 + x_3 = 2x_4$ and Abbott [2] improved this to $f(n) > 10^{\frac{n}{3} - c \log n}$ for some constant c and n sufficiently large. Theorem 2.3.1 may be used to improve this result to

$$f(n) > c\, 10^{\frac{n}{3}}$$

for some constant c.

Clearly estimates for $f(n)$ for many regular equations can be found in this manner. However the difficulty in determining $h(m)$ may be as difficult in general as determining $f(n)$ itself.

§2.4 A problem of Turán

Schur's theorem can be generalized in other directions. One such generalization is the following question raised by P. Turán [26] : If n and m are positive integers, denote by $f(m,n)$ the largest possible integer such that the integers $m, m+1, \ldots, m+f(m,n)$ can be partitioned into n sum-free sets. What can be said about $f(m,n)$?

It is clear that

$$f(1,n) = f(n) - 1$$

and since the integers $m, 2m, \ldots, m(f(n)+1)$ cannot be partitioned into n sum-free sets, that

$$f(m,n) \leq mf(n) - 1 . \tag{2.4.1}$$

Using (2.1.5) we have

$$f(m,n) \leq m[n!e] - m - 1 .$$

Turán considered the function $f(m,2)$ and proved that $f(m,2) = 4m - 1$. H.L. Abbott [2] observed that in fact we have equality in equation (2.4.1) for $m = 1,2,3$ and that

$$f(m,n+1) \geq 3f(m,n) + m + 2$$

and consequently

$$f(m,n) \geq \frac{m3^n - m - 2}{2} . \tag{2.4.2}$$

S. Znám has also studied the function $f(m,n)$, [29], but does not obtain any improvements on the results of Abbott. In [2] Abbott asks whether there exists a constant $c > 3$ such that

$$f(m,n) > mc^n$$

for all m and all n sufficiently large? We can now answer this question in the affirmative.

First we prove the following:

<u>Theorem 2.4.1</u> For any positive integer n, define $g(n)$ to be the largest positive integer such that the integers $1,2,\ldots,g(n)$ may be partitioned into n classes, none of which contain a solution of either of the equations

$$(2.4.3)\qquad \begin{gathered} x_1 + x_2 = x_3 \\ x_1 + x_2 + 1 = x_3 \quad . \end{gathered}$$

We will call such classes <u>strongly sum-free</u>. Then for any positive integer m

$$g(n+m) \geq 2f(m)g(n) + f(m) + g(n)$$

where $f(m)$ is the Schur function for the equation $x_1 + x_2 = x_3$.

<u>Proof</u>: Given a partitioning of $1,2,\ldots,f(m)$ into m sum-free classes $A_1,A_2,\ldots,A_m$, partition the integers $1,2,\ldots,2f(m) + 1$ into $m+1$ classes $B_1,B_2,\ldots,B_{m+1}$ as follows:

$$B_i = \{2a \mid a \in A_i\} \qquad i = 1,2,\ldots,m$$

$$B_{m+1} = \{1,3,5,\ldots,2f(m) + 1\} \quad .$$

The classes B_i , for $i = 1,2,\ldots,m$ are strongly sum-free and B_{m+1} is sum-free.

Partition the integers $1,2,\ldots,g(n)$ into n strongly sum-free classes $C_1,C_2,\ldots,C_n$. Construct $m+n$ classes D_j , $j = 1,2,\ldots,m+n$ as follows: For $j = 1,2,\ldots,m$

$$D_j = \{(2a-1)g(n)+a+b \mid 2a \in B_j, \quad b = 0,1,\ldots,g(n)\}$$

and for $j = 1,2,\ldots,n$

$$D_{m+j} = \{2ag(n)+a+b \mid 2a+1 \in B_{m+1}, \ b \in C_j\} \quad .$$

Then the classes $D_1, D_2, \ldots, D_{m+n}$ contain the integers $1, 2, \ldots, 2f(m)g(n) + f(m) + g(n)$ and it remains to be shown that they are strongly sum-free.

Suppose that for some j, $1 \leq j \leq m$, D_j is not strongly sum-free. Then either

$$(2.4.4) \qquad (2a_1-1)g(n)+a_1+b_1+(2a_2-1)g(n)+a_2+b_2 = (2a_3-1)g(n)+a_3+b_3$$

or

$$(2.4.5) \qquad (2a_1-1)g(n)+a_1+b_1+(2a_2-1)g(n)+a_2+b_2+1 = (2a_3-1)g(n)+a_3+b_3$$

where in each case $a_1, a_2, a_3 \in A_j$ and $0 \leq b_1, b_2, b_3 \leq f(n)$. Now (2.4.4) implies

$$(2.4.6) \qquad (2g(n)+1)(a_1+a_2-a_3) = g(n)+b_3-b_1-b_2 \quad .$$

Since A_j is sum-free, $a_1 + a_2 - a_3 \neq 0$. Therefore the left side of (2.4.6) is at least $2g(n)+1$ in absolute value, while the right side is at most $2g(n)$. Hence (2.4.4) cannot hold. A similar argument shows that (2.4.5) cannot hold and thus D_j is strongly sum-free for $j = 1, 2, \ldots, m$.

Now suppose some class D_{m+j}, $1 \leq j \leq n$ is not strongly sum-free. Then either

$$(2.4.7) \qquad 2a_1g(n)+a_1+b_1+2a_2g(n)+a_2+b_2 = 2a_3g(n)+a_3+b_3$$

or

$$(2.4.8) \qquad 2a_1g(n)+a_1+b_1+2a_2g(n)+a_2+b_2+1 = 2a_3g(n)+a_3+b_3$$

where in each case $2a_1+1,\ 2a_2+1,\ 2a_3+1 \in B_{m+1}$ and $b_1, b_2, b_3 \in C_j$. Now

(2.4.7) implies

$$(2g(n)+1)(a_1+a_2-a_3) = b_3-b_1-b_2 \quad . \tag{2.4.9}$$

But, since $b_1,b_2,b_3 \varepsilon C_j$, (2.4.9) implies $a_1 + a_2 - a_3 = 0$ and thus $b_1 + b_2 = b_3$ which contradicts the definition of $g(n)$. A similar argument shows that (2.4.8) cannot hold. Hence D_{m+j} is strongly sum-free for $j = 1,2,\ldots,n$ and the theorem is proved.

<u>Theorem 2.4.2</u> For any positive integers m and n

$$f(m,n) \geq mg(n) - 1 \quad .$$

<u>Proof</u>: Partition the integers $1,2,\ldots,g(n)$ into n strongly sum-free classes $C_1,C_2,\ldots,C_n$. Now partition the integers $m,m+1,\ldots,mg(n)+m-1$ into n classes $C_1', C_2', \ldots, C_n'$ by placing $am+b$ in C_i' whenever $a \varepsilon C_i$, where $a = 1,2,\ldots,g(n)$ and $b = 0,1,\ldots,m-1$.

Suppose from some j , $1 \leq j \leq n$, C_j' is not sum-free. Then we must have

$$a_1m+b_1 + a_2m+b_2 = a_3m+b_3 \tag{2.4.10}$$

where $a_1,a_2,a_3 \varepsilon C_j$ and $0 \leq b_1,b_2,b_3 \leq m-1$. Equation (2.4.10) implies

$$m(a_1+a_2-a_3) = b_3 - b_1 - b_2 \quad . \tag{2.4.11}$$

But, since C_j is strongly sum-free, the left hand side of (2.4.11) is either at least m or at most $-2m$. It now follows since $0 \leq b_1,b_2,b_3 \leq m-1$ that C_j' is sum-free and the theorem is proved.

We may now obtain a much stronger result than that given by (2.4.2).

Corollary 2.4.1 For any positive integers m and n

$$f(m,n) \geq m(3f(n-1) + 1) - 1 \quad .$$

Proof: Let $n = 1$ and $m = n-1$ in Theorem 2.4.1 and we have $g(n) \geq 3f(n-1) + 1$ and the result now follows from Theorem 2.4.2.

Corollary 2.4.2 For any positive integers m and n

$$f(m,n) > cm\, 89^{\frac{n}{4}}$$

for some absolute constant c .

Proof: This is an immediate consequence of Corollary 2.4.1 and Corollary 2.1.2.

§2.5 Some related questions

An analogous problem to that of sum-free sets is that of product free sets, i.e., what is the largest positive integer $\ell(n)$ such that the integers $2,3,\ldots,\ell(n)$ can be partitioned into n classes, no class containing a solution to the equation $x_1x_2 = x_3$? It is easy to see that

$$2^{\frac{3^n+1}{2} - 1} \leq \ell(n) \leq 2^{f(n)+1} - 1 \tag{2.5.1}$$

where $f(n)$ is the Schur function for sum-free sets.

If we partition the integers $2^1, 2^2, \ldots, 2^{g(n)}$ into n classes $C_1, C_2, \ldots, C_n$, where $g(n)$ is the function defined in Theorem 2.4.1, and place $2^k + j$ in class C_i whenever $2^k \in C_i$ and $j = 0, 1, \ldots, 2^k - 1$, then it is easy to see that

<u>Theorem 2.5.1</u> For any positive integer n

$$2^{g(n)+1} - 1 \leq \ell(n) \leq 2^{f(n)+1} - 1 \quad .$$

<u>Corollary 2.5.1</u> For any positive integer n

$$\ell(n) \geq 2^{3f(n-1)+1} - 1 \quad .$$

These results clearly are substantial improvements of (2.5.1).

We now consider an analogous problem in set theory: Given a positive integer n, what is the minimum number, $k(n)$, such that the 2^n subsets of a set S of n elements can be partitioned into $k(n)$ union-free classes?

Consider the following partitioning of the integers $1, 2, \ldots, n$:

$$C_1 = \{1, 3, 7, \ldots\}$$

$$C_i = \{2(i-1), 4(i-1)+1, 8(i-1)+3, \ldots\} \qquad i = 2, 3, \ldots, [\tfrac{n}{2}]+1 \quad .$$

If we now place all the subsets of S of order k in a class C_i' whenever $k \in C_i$ it is easy to see that

$$k(n) \leq [\tfrac{n}{2}] + 1 \quad . \tag{2.5.2}$$

On the other hand, D. Kleitman [18] has shown, for some constant c, that no union-free class can contain more than $c \frac{2^n}{\sqrt{n}}$ subsets of S. Therefore it follows that

$$k(n) > c\sqrt{n} \tag{2.5.3}$$

for some constant c.

At the present time we have not been able to improve either of these results even though one might expect $k(n)$ to be closer to (2.5.2) than to (2.5.3).

One can also raise similar questions about sum-free sets in Abelian groups. Let G be an Abelian group of order n and denote by $f(G)$ the least number of sum-free sets into which $G - \{e\}$ can be partitioned and denote by $f(n)$ the maximum of $f(G)$ where the maximum is taken over all Abelian groups of order n. Then the orginal Schur argument can be modified to give

$$f(n) > \frac{c \log n}{\log \log n}$$

for some constant c and all sufficiently large n.

We can prove that

$$f(n) < c_1 \log n$$

for some constant c_1 and all sufficiently large n. We have not been able to sharpen the bounds given above. In fact we cannot even evaluate $f(G)$ for Abelian groups of "small" order.

CHAPTER III

ON A PROPERTY OF FAMILIES OF SETS

§3.1. Property $\mathcal{B}$

A family $\mathcal{F}$ of sets is said to possess property $\mathcal{B}$ if there exists a set $B \subset \cup \mathcal{F}$ such that $B \cap F \neq \emptyset$ and $B \not\supset F$ for every $F \in \mathcal{F}$.

Several well known theorems are related in some sense to property $\mathcal{B}$. For example, a theorem of van der Waerden [27] states that to each positive integer $k \geq 3$ there corresponds a least positive integer $w(k)$, such that if the integers $1,2,\ldots,w(k)$ are partitioned in an arbitrary manner into two classes, at least one class contains an arithmetic progression of k terms. P. Erdös [10] pointed out that van der Waerden's theorem can be formulated as follows: Let $\mathcal{F}_{k,w}$ denote the family of all arithmetic progressions of k terms contained in the interval $[1,w]$. Then there exists a least positive integer $w(k)$ such that if $w \geq w(k)$, then $\mathcal{F}_{k,w}$ does not possess property β.

Ramsey's theorem can be used to prove the following (see [2]): Let $s \geq R_2(k,r)$ and let S be a set of s elements. Let K be a subset of S with k elements and let F be the set whose elements are the $\binom{k}{r}$ subsets of K with r elements. Let $\mathcal{F}_{k,r}$ be the family of all possible sets constructed in this way. Then $\mathcal{F}_{k,r}$ does not possess property $\mathcal{B}$.

There have been several papers written about families of sets which do or do not possess property $\mathcal{B}$. The first papers on the subject were devoted primarily to the study of infinite families of infinite sets. However, as was pointed out by Erdös and Hajnal [14] several interesting questions can be asked about finite families of finite sets.

Erdös and Hajnal posed the following problem: What is the smallest integer $m(n)$ for which there exists a family, $\mathcal{F}_n$, of sets $A_1, A_2, \ldots, A_{m(n)}$ such that $|A_i| = n$ for $i = 1,2,\ldots,m(n)$ and which does not possess property $\mathcal{B}$? They observed that $m(1) = 1$, $m(2) = 3$ and $m(3) = 7$. That $m(2) \leq 3$ and $m(3) \leq 7$ follows from the fact that the families

$$\mathcal{F}_2 = \{(1,2),\ (2,3),\ (1,3)\}$$

and

$$\mathcal{F}_3 = \{(1,2,3,),(1,4,5),(1,6,7),(2,4,6),(2,5,7),(3,4,7),(3,5,6)\}$$

do not possess property $\mathcal{B}$. By trial and error one can easily show that $m(2) > 2$ and $m(3) > 6$. The value of $m(n)$ for $n \geq 4$ is not known and appears very difficult to determine.

H. L. Abbott and L. Moser [5] showed that

$$m(ab) \leq m(a)m(b)^a \tag{3.1.1}$$

for all integers a and b and from this deduced that for every $\varepsilon > 0$

$$m(n) = 0(\sqrt{7} + \varepsilon)^n , \tag{3.1.2}$$

and that

$$\lim_{n \to \infty} m(n)^{\frac{1}{n}}$$

exists. Erdös [11] proved that, for all $n \geq 2$

$$m(n) > 2^{n-1}$$

and by more complicated arguments he was able to prove that

$$m(n) > (1-\varepsilon)2^n \log 2$$

for every $\varepsilon > 0$ and $n \geq n_o(\varepsilon)$. The best lower bound that has been obtained up to the present time is the following due to W. Schmidt [24] :

$$m(n) > 2^n \left(\frac{n}{n+4}\right) .$$

More recently Erdös [12] proved that for all n

$$m(n) \leq n^2 \; 2^{n+1}$$

and that for every $\varepsilon > 0$ and n sufficiently large,

$$m(n) \leq (1-\varepsilon)n^2 \; 2^{n-1} \log 2 \quad .$$

It follows easily from these results that

$$\lim_{n \to \infty} m(n)^{\frac{1}{n}} = 2 \quad .$$

In [12] Erdös raised the following question: Let $N \geq 2n-1$ and denote by $m_N(n)$ the smallest positive integer for which there exists a family $\mathcal{F}$ of subsets $A_1, A_2, \ldots, A_{m_N(n)}$ of a set of N elements

such that $|A_i| = n$ for $i = 1,2,\ldots,m_N(n)$, and which does not possess property $\mathcal{B}$. It is not difficult to show that

$$m_{2n-1}(n) = \binom{2n-1}{n} ,$$

and it is clear that if N is sufficiently large

$$m_N(n) = m(n) .$$

In [13] Erdös proved that

$$m_{2N-1}(n) \geq m_{2N}(n) \geq 2^{n-1} \prod_{i=o}^{n} (1 + \frac{i}{2N-2i})$$

and

$$m_{2N+1}(n) \leq m_{2N}(n) \leq N2^n \prod_{i=o}^{n-1} (1 - \frac{i}{2n-i})^{-1} .$$

This problem suggests the following question raised by Erdös in [13]: What can be said about $m_N(n)$ if we impose the condition that $|\cup\mathcal{F}| = N$ and if $\mathcal{F}'$ is a proper subfamily of $\mathcal{F}$ and $|\cup\mathcal{F}'| < N$ then $\mathcal{F}'$ has property $\mathcal{B}$? Here there are two questions; first, do such families exist for given integers N and n and secondly, whenever such families exist, what can be said about their size? It is to a slight modification of this problem that we devote this chapter.

§3.2 A problem of P. Erdös

In [13] Erdös posed the following problem: Let n and N be positive integers, $n \geq 2$ and $N \geq 2n-1$ and let S be a set of

N elements. What is the least integer $m'_N(n)$, (provided such an integer exists), for which there exists a family $\mathcal{F}$ of $m'_N(n)$ subsets of S satisfying

(a) $|F| = n$ for each $F \in \mathcal{F}$

(b) $\bigcup \mathcal{F} = S$

(c) $\mathcal{F}$ does not have property $\mathcal{B}$

(d) If $\mathcal{F}' \subset \mathcal{F}$ and $|\bigcup \mathcal{F}| < N$ then $\mathcal{F}'$ has property $\mathcal{B}$?

Erdös pointed out that $m'_{2t+1}(2) = 2t+1$ and that $m'_N(2)$ does not exist if N is even. This is just a restatement of the fact that the only critical three chromatic graphs are the odd circuits. We shall prove that $m'_N(n)$ exists for all $n \geq 3$, $N \geq 2n-1$, and obtain some upper bounds. However, instead of studying $m'_N(n)$ we shall consider the function $m^*_N(n)$ which is defined in the same manner as $m'_N(n)$ except that (d) is replace by

(e) If $\mathcal{F}'$ is a proper subfamily of $\mathcal{F}$ then $\mathcal{F}'$ has property $\mathcal{B}$.

It is clear that the existence of $m^*_N(n)$ implies the existence of $m'_N(n)$ and in fact $m'_N(n) \leq m^*_N(n)$. Further we note that

$$m(n) = \min_N m^*_N(n) .$$

<u>Theorem 3.2.1</u> If $m^*_N(a)$ exists, then for every positive integer b , $m^*_{N+a+2b-1}(a+b)$ exists.

<u>Proof:</u> Let $\mathcal{G}$ be a family of $m^*_N(a)$ sets satisfying (a), (b), (c) and (e) . Let T be a set with a+2b-1 elements. We assume that T is

disjoint from every member of $\mathcal{G}$. Let $\mathcal{H}$ be the family of b-subsets of T and let $\mathcal{L}$ be the family of a+b-subsets of T. Let

$$\mathcal{F} = \{F \mid F = H \cup G,\ H \in \mathcal{H},\ G \in \mathcal{G} \text{ or } F = L,\ L \in \mathcal{L}\} .$$

It is clear that each member of $\mathcal{F}$ has $a+b$ elements and that $|\cup\mathcal{F}| = N+a+2b-1$. We need to show that $\mathcal{F}$ satisfies (c) and (e). Suppose $\mathcal{F}$ has property $\mathcal{B}$. Then there exists a set $B \subset \cup\mathcal{F}$ such that $0 < |B \cap F| < a+b$ for every $F \in \mathcal{F}$. Since $\mathcal{G}$ does not have property $\mathcal{B}$, either there exists a set $G_1 \in \mathcal{G}$ such that $B \cap G_1 = \emptyset$ or there exists a set $G_2 \in \mathcal{G}$ such that $B \supset G_2$. However $B \cap G_1 = \emptyset$ implies $B \cap H \neq \emptyset$ for each $H \in \mathcal{H}$. Hence $|B \cap T| \geq a+b$ and $B \supset L$ for some $L \in \mathcal{L}$. This is a contradiction and therefore $B \cap G_1 = \emptyset$ is impossible. Also $B \cap L \neq \emptyset$ for all $L \in \mathcal{L}$ implies $|B \cap T| \geq b$ and hence $B \supset H_1$ for some $H_1 \in \mathcal{H}$. Thus $B \supset G_2$ is also impossible since it would imply $B \supset H_1 \cup G_2$. Hence $\mathcal{F}$ does not have property $\mathcal{B}$.

Let $\mathcal{F}'$ be a proper subfamily of $\mathcal{F}$. We need to show that there exists a set B such that $0 < |B \cap F| < a+b$ for all $F \in \mathcal{F}'$. Let $A \in \mathcal{F} - \mathcal{F}'$. Suppose first that $A = G_1 \cup H_1$, $G_1 \in \mathcal{G}$, $H_1 \in \mathcal{H}$. Since $\mathcal{G}$ satisfies (e), there exists a set $B_1 \subset \cup\mathcal{G}$ such that $0 < |B_1 \cap G| < a$ for all $G \in \mathcal{G} - \{G_1\}$. Set $B = H_1 \cup B_1$. Then clearly $0 < |B \cap F| < a+b$ for all $F \in \mathcal{F}'$. The only other possibility is that $A \in \mathcal{L}$. Then one can take $B = A$. It follows that $\mathcal{F}'$ has property $\mathcal{B}$ and the proof of Theorem 3.2.1 is complete.

<u>Corollary 3.2.1</u> $m_N^*(n)$ exists for all $n \geq 3$ and $N \geq 2n$, N even.

Proof: As was pointed out by Erdös, $m^*_{2t+1}(2)$ exists for all $t \geq 1$. If we take $a = 2$, $N = 2t+1$ in Theorem 3.2.1 we see that $m^*_{2t+2b+2}(b+2)$ exists for all $b \geq 1$, $t \geq 1$. This clearly implies the corollary.

Theorem 3.2.2 $m^*_{2t+1}(3)$ exists for all $t \geq 2$.

Proof: Let $\mathcal{G}$ be a family of $m^*_{2t-1}(2)$ 2-subsets of a set S of $2t-1$ elements satisfying (a), (b), (c) and (e). We may assume $S = \{1,2,\ldots,2t-1\}$ and $\mathcal{G} = \{(1,2),(2,3),(3,4),\ldots,(i,i+1),\ldots,(2t-2,2t-1),(1,2t-1)\}$, that is, the sets in $\mathcal{G}$ form the edges of an odd circuit. Let $\mathcal{F}$ be the family consisting of the following sets: $G \cup \{a\}$, $G \cup \{b\}$, $\{a,b,1\}$, $\{a,b,2\}$, $\{a,b,3\}$, $\{1,2,3\}$, where $G \in \mathcal{G}$ and $a,b \notin S$. To prove the theorem we must show that $\mathcal{F}$ satisfies (c) and (e).

Let $B \subset \cup\mathcal{F}$ have non-empty intersection with each member of $\mathcal{F}$. If $a \in B$ and $b \in B$ then, since $B \cap \{1,2,3\} \neq \emptyset$, B must contain one of $\{a,b,1\}$, $\{a,b,2\}$ or $\{a,b,3\}$. If $a \in B$ and $b \notin B$ then $B \cap G \neq \emptyset$ for each $G \in \mathcal{G}$. Since $\mathcal{G}$ does not have property $\mathcal{B}$, $B \supset G_1$ for some $G_1 \in \mathcal{G}$. Hence $B \supset G_1 \cup \{a\}$. Finally if $a \notin B$ and $b \notin B$ then $B \supset \{1,2,3\}$. Thus $\mathcal{F}$ does not have property $\mathcal{B}$.

Let $\mathcal{F}'$ be a proper subfamily of $\mathcal{F}$ and let $A \in \mathcal{F} - \mathcal{F}'$. We need to show that there exists a set B such that $0 < |B \cap F| < 3$ for all $F \in \mathcal{F}'$. If $A = G_1 \cup \{a\}$, $G_1 \in \mathcal{G}$, then since $\mathcal{G}$ satisfies (e), there exists a set $B_1 \subset \cup\mathcal{G}$ such that $|B_1 \cap G| = 1$ for each $G \in \mathcal{G} - \{G_1\}$ and $B_1 \cap G_1 = \emptyset$. Since at least one but not all of 1,2,3 belong to B_1 we may take $B = B_1 \cup \{b\}$. Then $0 < |B \cap F| < 3$

for all $F \in \mathcal{F}'$. The case $A = G_1 \cup \{b\}$ can be disposed of in the same way. If A is one of $\{a,b,1\}$, $\{a,b,2\}$ or $\{a,b,3\}$ we may choose $B = A$. Finally if $A = \{1,2,3\}$ we may take $B = \{a,b\}$. It follows that $\mathcal{F}'$ has property $\mathcal{B}$.

<u>Corollary 3.2.2</u> $m_N^*(n)$ exists for $n \geq 3$ and $N \geq 2n-1$, N odd.

<u>Proof:</u> The case $n = 3$ has been taken care of in Theorem 3.2.2. For $n > 3$ take $a = 3$, $N = 2t+1$ in Theorem 3.2.1. This shows that $m^*_{2t+2b+3}(b+3)$ exists for $b \geq 1$, $t \geq 2$ and hence Corollary 3.2.2 holds.

From the above results we obtain that $m_N^*(n)$ exists for all $n \geq 3$, $N \geq 2n-1$ and

$$m_N^*(n) \leq \begin{cases} (N-2n+3)\binom{2n-3}{n-2} + \binom{2n-3}{n}, & \text{if } N \text{ is even, } n \geq 3 \\ 2(N-2n+4)\binom{2n-4}{n-3} + \binom{2n-4}{n}, & \text{if } N \text{ is odd, } n \geq 4 . \end{cases}$$

<u>Theorem 3.2.3</u> If $K \geq 2k-1$ and $L \geq 2\ell-1$, then

$$m^*_{KL}(k\ell) \leq m_K^*(k)(m_L^*(\ell))^k$$

provided $m_K^*(k)$ and $m_L^*(\ell)$ exist.

<u>Proof:</u> Let $\mathcal{F}_k = \{A_1,A_2,\ldots,A_{m_K^*(k)}\}$ be a family of sets satisfying definition of $m_K^*(k)$. Let $\bigcup_{i=1}^{m_K^*(k)} A_i = \{x_1,x_2,\ldots,x_K\}$ and for $j = 1,2,\ldots,K$ let $\mathcal{F}_{j,\ell} = \{B_1^j,B_2^j,\ldots,B^j_{m_L^*(\ell)}\}$ be families of sets

satisfying the definition of $m_L^*(\ell)$. We assume that the sets in $\mathcal{F}_{j,\ell}$ are disjoint from the sets in $\mathcal{F}_{m,\ell}$ if $j \neq m$. Choose $A_i = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \in \mathcal{F}_k$ and from each of the families

$\mathcal{F}_{i_1,\ell}, \mathcal{F}_{i_2,\ell}, \ldots, \mathcal{F}_{i_k,\ell}$ pick a set B . The union of these B's is a set consisting of $k\ell$ elements. We shall say that such sets are generated by the set A_i . Let $\mathcal{F}$ be the family of all possible sets constructed in this manner. Clearly there are $m_K^*(k)(m_L^*(\ell))^k$ such sets; $|\cup \mathcal{F}| = KL$. Suppose $\mathcal{F}$ has property $\mathcal{B}$, i.e. there exists a set $B \subset \cup \mathcal{F}$ such that $B \cap F \neq \emptyset$ and $B \not\supset F$ for each $F \in \mathcal{F}$. There are two cases to be considered.

Case 1. There exists $A_i = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \in \mathcal{F}_k$ such that B has non-empty intersection with each member of each of the families $\mathcal{F}_{i_1,\ell}, \mathcal{F}_{i_2,\ell}, \ldots, \mathcal{F}_{i_k,\ell}$. Then B contains at least one member of each of these families and hence contains a member of $\mathcal{F}$.

Case 2. In each $A_i \in \mathcal{F}_k$ there is an element which we denote by $x_{A_i'}$ such that B has empty intersection with one of the sets in $\mathcal{F}_{A_i',\ell}$. Let $T = \{x_{A_i'} \mid i = 1,2,\ldots,m_K^*(k)\}$. Then $T \cap A \neq \emptyset$ for each $A \in \mathcal{F}_k$. Thus $T \supset A_j = \{x_{j_1}, x_{j_2}, \ldots, x_{j_k}\}$ for some j , $1 \leq j \leq m_k^*(k)$. It follows that B has empty intersection with at least one of the sets in each of the families $\mathcal{F}_{j_1,\ell}, \mathcal{F}_{j_2,\ell}, \ldots, \mathcal{F}_{j_k,\ell}$ and hence has empty intersection with at least one $F \in \mathcal{F}$. This contradicts the assumption $B \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. It follows that $\mathcal{F}$ does not have property $\mathcal{B}$.

It remains to show that any proper subfamily $\mathcal{F}'$ of $\mathcal{F}$ has property $\mathcal{B}$. Let $A \in \mathcal{F} - \mathcal{F}'$ and assume that A is generated by A_i for some i, $1 \leq i \leq m^*_K(k)$. By the definition of $m^*_K(k)$ there exists a set B' such that $B' \cap A_i = \emptyset$, $B' \cap A_j \neq \emptyset$ and $B' \not\supset A_j$ for all $j \neq i$. Let $B' = \{x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_t}\}$. For each of the families $\mathcal{F}_{\lambda_{j,\ell}}$, $j = 1, 2, \ldots, t$, choose a set B_{λ_j} such that $B_{\lambda_j} \cap F \neq \emptyset$ for all $F \in \mathcal{F}_{\lambda_{j,\ell}}$. Then if $B_1 = \cup B_{i_j}$, $B_1 \cap F \neq \emptyset$ and $B_1 \not\supset F$ for all $F \in \mathcal{F}$ not generated by A_i. Now consider those $F \in \mathcal{F}$ generated by A_i. Since $A = \bigcup_{j=1}^{k} F_{i_{j,\ell}}$ for some $F_{i_{j,\ell}} \in \mathcal{F}_{i_{j,\ell}}$, $j = 1, 2, \ldots, k$, there exists sets B'_{i_j}, $j = 1, 2, \ldots, k$, such that $B'_{i_j} \cap F \neq \emptyset$ and $B'_{i_j} \not\supset F$ for all $F \in \mathcal{F}_{i_{j,\ell}} - \{F_{i_{j,\ell}}\}$ and $B'_{i_j} \cap A = \emptyset$. Then $B_2 = \cup B'_{i_j}$ is such that B_2 has non-empty intersection with every set generated by A_i except the set A, and B_2 contains none of these sets. Clearly $B_1 \cap B_2 = \emptyset$ and hence if we choose $B = B_1 \cup B_2$, the set B shows that $\mathcal{F}'$ has property $\mathcal{B}$.

The first half of the above proof is essentially that used by Abbott and Moser to obtain the inequality (3.1.1). This inequality can readily be deduced from Theorem 3.2.3 as a consequence of the fact that $m(n) = \min_N m^*_N(n)$ and from the existence theorems already proven.

<u>Theorem 3.2.4</u> $m^*_{N+2n}(n) \leq nm^*_N(n-2)+2^{n-1}$ if n is odd.

<u>Proof:</u> Let $\mathcal{G}$ be a family of $m^*_N(n-2)$ sets satisfying (a), (b), (c) and (e). Let $F_i = \{2i-1, 2i\}$ and let $\mathcal{H} = \{F_i \mid i = 1,2,\ldots,n\}$. Let $\mathcal{L}$ be the family consisting of all sets of the form $\{a_1, a_2, \ldots, a_n\}$

where $a_i \in F_i$ and $a_i = 2i-1$ for an even number of values of i. We may assume that $G \cap \{1,2,\ldots,2n\} = \emptyset$ for all $G \in \mathcal{G}$. Finally let

$$\mathcal{F} = \{F \mid F = G \cup H,\ G \in \mathcal{G},\ H \in \mathcal{H} \text{ or } F = L,\ L \in \mathcal{L}\}.$$

It is clear that the number of sets in $\mathcal{F}$ is $n\, m_N^*(n-2) + 2^{n-1}$ and that $|\bigcup \mathcal{F}| = N+2n$. It remains to be shown that $\mathcal{F}$ satisfies conditions (c) and (e).

Let B be any set such that $B \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. To show that $\mathcal{F}$ does not have property $\mathcal{B}$ we must show that B contains a member of $\mathcal{F}$.

<u>Case 1</u>. For some i and j, $B \supset F_i$ and $B \cap F_j = \emptyset$.

$B \cap F_j = \emptyset$ implies that $B \cap G \neq \emptyset$ for each $G \in \mathcal{G}$. Since $\mathcal{G}$ does not have property $\mathcal{B}$, $B \supset G_1$ for some $G_1 \in \mathcal{G}$. Hence $B \supset G_1 \cup F_i$, that is B contains a member of $\mathcal{F}$.

<u>Case 2</u>. For some i, $B \supset F_i$, $|B \cap F_j| \geq 1$ for all j.

We may assume without loss of generality that $B \supset F_1, F_2, \ldots, F_t$ and $|B \cap F_j| = 1$ for $j = t+1, t+2, \ldots, n$. In fact we may assume $B \cap F_j = \{2j-1\}$ for $j = t+1, t+2, \ldots, t+r$ and $B \cap F_j = \{2j\}$ for $j = t+r+1, t+r+2, \ldots, n$. If r is even then B contains $\{2, 4, \ldots, 2t, 2t+1, \ldots, 2t+2r-1, 2t+2r+2, \ldots, 2n\}$ and if r is odd, B contains $\{1, 4, 6, \ldots, 2t, 2t+1, \ldots, 2t+2r-1, 2t+2r+2, \ldots, 2n\}$ so that in any case B contains a member of $\mathcal{F}$.

<u>Case 3</u>. $|B \cap F_i| \leq 1$ for all i, $B \cap F_j = \emptyset$ for some j.

In this case $\overline{B}$, the complement of B with respect to $\{1,2,\ldots,2n\}$, satisfies the conditions of Case 2. Hence $\overline{B}$ contains a member of $\mathcal{F}$ and thus B is disjoint from a member of $\mathcal{F}$ which contradicts our assumption that $B \cap F \neq \emptyset$ for each $F \in \mathcal{F}$.

<u>Case 4</u>. $|B \cap F_i| = 1$ for all i.

We may assume $B \cap F_i = \{2i-1\}$ for $i = 1,2,\ldots,t$ and $B \cap F_i = \{2i\}$ for $i = t+1,t+2,\ldots,n$. If t is even then B contains $\{1,3,\ldots,2t-1,2t+2,\ldots,2n\}$, i.e. B contains a member of $\mathcal{F}$. If t is odd, then since n is odd B is disjoint from $\{2,4,\ldots,2t,2t+1,\ldots,2n-1\}$, i.e. B is disjoint from a member of $\mathcal{F}$ and this is impossible.

Since there are no other possibilities, $\mathcal{F}$ does not have property β.

Let $\mathcal{F}'$ be a proper subfamily of $\mathcal{F}$. We must show that there exists a set B such $0 < |B \cap F| < n$ for all $F \in \mathcal{F}'$. Let $A \in \mathcal{F} - \mathcal{F}'$. Suppose first that $A = G_1 \cup F_i$, $G_1 \in \mathcal{G}$. Since $\mathcal{G}$ satisfies (e), there exists a set $B_1 \subset \cup \mathcal{G}$ such that $0 < |B_1 \cap G| < n-2$ for all $G \in \mathcal{G} - \{G_1\}$. Moreover we may assume $B_1 \supset G_1$ (for if $B_1 \not\supset G_1$ then we must have $B_1 \cap G_1 = \emptyset$ and instead of choosing B_1 we choose $\overline{B}_1$, the complement of B_1 with respect to $\mathcal{G}$). Let $B = B_1 \cup F_i$. Then it is easy to see that $0 < |B \cap F| < n$ for all $F \in \mathcal{F}'$. The only other possibility is that $A \in \mathcal{L}$. In this case we take $B = A$. Then clearly $|B \cap F| < n$ for all $F \in \mathcal{F}'$. Moreover $B \cap F \neq \emptyset$ since

$B \cap F = \emptyset$ implies $F = \overline{A}$, the complement of A with respect to $\{1,2,\ldots,2n\}$. But A contains an even number of odd elements and since n is odd, $\overline{A}$ must contain an odd number of odd elements and hence $\overline{A} \notin \mathcal{F}$. Therefore $0 < |B \cap F| < n$ for all $F \in \mathcal{F}'$ and $\mathcal{F}'$ has property $\mathcal{B}$. This completes the proof of the theorem.

<u>Corollary 3.2.4</u> $m(5) \le 51$ and $m(7) \le 421$.

<u>Proof:</u> It is easy to see that $m_7^*(3) = 7$, hence

$$m_{17}^*(5) \le 5.7 + 16 = 51$$

and

$$m_{31}^*(7) \le 7.51 + 64 = 421 .$$

The best previous results known are $m(5) \le 126$ and $m(7) \le 708$.

<u>Theorem 3.2.5</u> $m_{N+2n}^*(n) \le n\, m_N^*(n-2) + 2^{n-1} + 2^{n-2}$ if n is even.

<u>Proof:</u> Let $\mathcal{G}$ be a family of $m_N^*(n-2)$ sets satisfying (a), (b), (c) and (e). Let $F_i = \{2i-1,2i\}$ and let $\mathcal{H} = \{F_i \mid i = 1,2,\ldots,n\}$. Let $\mathcal{L}_1$ be the family of all sets of the form $\{1,a_2,a_3,\ldots,a_n\}$ where $a_i \in F_i$. Let $\mathcal{L}_2$ be the family of all sets of the form $\{2,a_2,a_3,\ldots,a_n\}$ where $a_i \in F_i$ and $a_i = \{2i\}$ for an even number of values of i. We may assume that $G \cap \{1,2,\ldots,2n\} = \emptyset$ for all $G \in \mathcal{G}$. Finally let

$$\mathcal{F} = \{F \mid F = G \cup H,\ G \in \mathcal{G},\ H \in \mathcal{H} \text{ or } F = L,\ L \in \mathcal{L}_1 \text{ or } \mathcal{L}_2\} .$$

It is clear that the number of sets in $\mathcal{F}$ is $n\, m^*_N(n-2) + 2^{n-1} + 2^{n-2}$ and that $|\cup\mathcal{F}| = N+2n$. It remains to show that $\mathcal{F}$ satisfies conditions (c) and (e).

Let B be any set such that $B \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. To show that $\mathcal{F}$ does not have property $\mathcal{B}$ we must show that B contains a member of $\mathcal{F}$.

Case 1. For some i and j, $B \supset F_i$ and $B \cap F_j = \emptyset$.

$B \cap F_i = \emptyset$ implies that $B \cap G \neq \emptyset$ for each $G \in \mathcal{G}$. Since $\mathcal{G}$ does not have property $\mathcal{B}$, $B \supset G_1$ for some $G_1 \in \mathcal{G}$. Hence $B \supset G_1 \cup F_i$, that is, B contains a member of $\mathcal{F}$.

Case 2. For some i, $B \supset F_i$, and for all j, $|B \cap F_j| \geq 1$.

If $1 \in B$ then B contains a member of $\mathcal{L}_1$. Therefore we assume $1 \notin B$ and hence $2 \in B$ and $i \neq 1$. Then B contains a set of the form $\{2,a_2,a_3,\ldots,a_n\}$ where $a_j \in F_j$. If $a_j = 2j$ for an even number of values of j then B contains a member of $\mathcal{L}_2$. If $a_j = 2j$ for an odd number of values of j and if $a_i = 2i$ (respectively $2i-1$), replacing $2i$ by $2i-1$ (respectively replacing $2i-1$ by $2i$) we now have that $a_j = 2j$ for an even number of values of j and again B contains a member of $\mathcal{L}_2$.

Case 3. $|B \cap F_i| \leq 1$ for all i, and $B \cap F_j = \emptyset$ for some j.

In this case $\overline{B}$, the complement of B with respect to $\{1,2,\ldots,2n\}$ satisfies the conditions of Case 2. Hence $\overline{B}$ contains a

member of $\mathcal{F}$ and thus B is disjoint from a member of $\mathcal{F}$ which contradicts our original assumption that $B \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.

Case 4. $|B \cap F_i| = 1$ for all i .

As in Case 2, if $1 \in B$ then B contains a member of $\mathcal{F}$. Therefore we may assume that B contains a set of the form $\{2,a_2,a_3,\ldots,a_n\}$ where $a_i \in F_i$. If $a_i = 2i$ for an even number of values of i then B contains a member of $\mathcal{L}_2$. If $a_i = 2i$ for an odd number of values of i then B is disjoint from the set $\{1\} \cup (F_2-\{a_2\}) \cup \ldots \cup (F_n-\{a_n\})$ which belongs to $\mathcal{L}_1$ and we again have a contradiction.

Since there are no other possibilities, $\mathcal{F}$ does not have property $\mathcal{B}$.

Let $\mathcal{F}'$ be a proper subfamily of $\mathcal{F}$, we must show that there exists a set B such that $0 < |B \cap F| < n$ for all $F \in \mathcal{F}'$. Let $A \in \mathcal{F} - \mathcal{F}'$. Suppose first that $A = G_1 \cup F_i$, $G_1 \in \mathcal{G}$. Since $\mathcal{G}$ satisfies (e), there exists a set $B_1 \subset \cup \mathcal{G}$ such that $0 < |B_1 \cap G| < n-2$ for all $G \in \mathcal{G} - \{G_1\}$. Moreover we may assume $B_1 \supset G_1$ as in the proof of Theorem 3.2.4. Let $B = B_1 \cup F_i$. Then it is easy to see that $0 < |B \cap F| < n$ for all $F \in \mathcal{F}'$.

Now suppose $A = L_1$, $L_1 \in \mathcal{L}_1$ and $A = \{1,a_2,a_3,\ldots,a_n\}$ where $a_i \in F_i$. If $a_i = 2i$ for an odd number of values of i , let $B = A \cup \{2\}$. Then $B \cap F \neq \emptyset$ for all $F \in \mathcal{F}'$. If $B \supset F$ for some $F \in \mathcal{F}'$ then F belongs to $\mathcal{L}_2$. Then $B - \{1\}$ contains F . But $|B - \{1\}| = n$ and contains an odd number of elements $2i$. Hence we have a contradiction. If $a_i = 2i$ for an even number of values of i ,

let $B = A$. Then clearly $|B \cap F| < n$ for all $F \in \mathcal{F}'$. Moreover $B \cap F \neq \emptyset$ for all $F \in \mathcal{F}'$, for $B \cap F = \emptyset$ implies that F is $\overline{A}$, the complement of A with respect to $\{1,2,\ldots,2n\}$. But $\overline{A}$ has an odd number of elements $2i$, $i > 1$ and no such sets belong to $\mathcal{F}$.

Finally, suppose $A \in \mathcal{L}_2$. If $3 \in A$ let $B = A \cup \{4\}$ and if $4 \in A$ let $B = A \cup \{3\}$. Then clearly $B \cap F \neq \emptyset$ for all $F \in \mathcal{F}'$. If $B \supset F$ for some $F \in \mathcal{F}'$ then F must belong to $\mathcal{L}_2$, but then $F = A$, a contradiction, since $A \notin \mathcal{F}'$. Hence $\mathcal{F}'$ has property $\mathcal{B}$. This completes the proof of the theorem.

<u>Corollary 3.2.5</u> $m(4) \leq 24$.

<u>Proof</u>: Since $m_3^*(2) = 3$, we have by the above theorem

$$m_{11}^*(4) \leq 4.3 + 8 + 4 = 24 .$$

The best known previous result was $m(4) \leq 26$.

The final theorem we will state in this chapter is quite weak for large values of n and N , however it is useful to estimate $m_N^*(n)$ for small values of these variables.

Let $\mathcal{G}$ be a family of $m_N^*(n-1)$ sets satisfying (a), (b), (c) and (e) . Let $T = \{N+1,N+2,\ldots,N+n-1\}$ and $S = \{N+n-1,N+n,\ldots,N+2n-1\}$.

Now let

$$\mathcal{F} = \{F \mid F = G \,\dot{\cup}\, \{t\},\ G \in \mathcal{G},\ t \in T,\ F = T \,\dot{\cup}\, \{s\},\ s \in S \text{ or } F = S\}\ .$$

Using this construction, and by successively replacing elements of S by elements from $\cup \mathcal{G}$, we can prove the following theorem by similar methods to those used previously:

<u>Theorem 3.2.6</u> For $n \geq 3$ and $i = 0,1,\ldots,n-1$

$$m^*_{N+n+i}(n) \leq (n-1)\ m^*_N(n-1)+n+1\ .$$

Theorem 3.2.7 is useful in estimating $m^*_N(n)$ for small values of n and N.

In conclusion we mention some problems related to $m^*_N(n)$ which we have been unable to settle. For fixed n, for what range of values of N is $m^*_N(n)$ decreasing or increasing? Secondly for fixed n, does does $\lim\limits_{N \to \infty} \frac{m^*_N(n)}{N}$ exist? It follows from our results that there exists constants a_n and b_n such that for all N sufficiently large

$$a_n < \frac{m^*_N(n)}{N} < b_n\ .$$

Since for $n \geq 3$, $m^*_N(n)$ exists for all $N \geq 2n-1$, it is natural to ask the following question: Does there exist a countably infinite family $\mathcal{F}$ of sets, each set with n elements such that $\mathcal{F}$ does not have property $\mathcal{B}$ but every proper subfamily does? That such families do not exist was proved by Erdös and Hajnal [14]. In fact they pointed out that if $\mathcal{F}$ is a countably infinite family of sets, each set with n elements, and if every finite subfamily of $\mathcal{F}$ has

property $\mathcal{B}$, then so does $\mathcal{F}$. This result may also be proved using König's 'lemma of infinity' [19] .

The following is a table of estimates of $m_N^*(n)$ for certain values of n and N obtained from the theorems of this chapter:

N \ n	2	3	4	5
3	3			
4	-			
5	5	10		
6	-	10		
7	7	7	35	
8	-	10	35	
9	9	14	27	126
10	-	14	26	126
11	11	18	24	156
12	-	18	26	146
13	13	22	26	146
14	-	22	26	110
15	15	26	35	66
16	-	26	47	66
17	17	30	47	51
18	-	30	59	66
19	19	34	56	86
20	-	34	71	86

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